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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

Ceux qui aiment l'Analyse, verront avec plaisir la Méchanique en devenir une nouvelle branche ... LAGRANGE

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The Theory of Matrix Polynomials and its Application to the Mechanics of Isotropic Continua

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Abstract

In this paper we show that a symmetric isotropic matrix polynomial in any number of symmetric 3×3 matrices can be expressed as a symmetric isotropic matrix polynomial, in which each of the matrix products is formed from at most six matrices and has one of a certain number of forms which are explicitly given. The significance of these results in the mechanics of isotropic continua is indicated.

1. Introduction

We consider a body to undergo deformation described in a rectangular Cartesian coordinate system x by

$$x_i = x_i(X_j, t),$$

where x_i and X_i are the coordinates in the system x of a generic particle of the body at arbitrary time t and a standard reference time respectively. We assume that the stress components t_{ij} at the point x_i and time t are single-valued functions of the deformation gradients $\partial x_p/\partial X_q$ and of the gradients of the velocity $v_p^{(1)}$, the acceleration $v_p^{(2)}, \ldots,$ the $(n-1)$th acceleration $v_p^{(n)}$, at that point and time; *i.e.* the constitutive equation takes the form

$$t_{ij} = t_{ij}\left(\frac{\partial x_p}{\partial X_q}, \frac{\partial v_p^{(1)}}{\partial x_q}, \ldots, \frac{\partial v_p^{(n-1)}}{\partial x_q}\right). \tag{1.1}$$

It has been shown [1] that if, further, the material is isotropic at the standard reference time, the stress components t_{ij} must be expressible as single-valued functions of the quantities C_{pq} and $A^{(r)}_{pq}$ $(r=1, 2, \ldots, n)$ defined by

$$C_{pq} = \frac{\partial x_p}{\partial X_m} \frac{\partial x_q}{\partial X_m}, \qquad A^{(1)}_{pq} = \frac{\partial v^{(1)}_p}{\partial x_q} + \frac{\partial v^{(1)}_q}{\partial x_p}$$

and

$$A^{(r)}_{pq} = \frac{D A^{(r)}_{pq}}{Dt} + A^{(r-1)}_{pm} \frac{\partial v^{(1)}_m}{\partial x_q} + A^{(r-1)}_{mq} \frac{\partial v^{(1)}_m}{\partial x_p}, \tag{1.2}$$

where D/Dt denotes the material time derivative; *i.e.* the constitutive equation takes the form

$$t_{ij} = t_{ij}(C_{pq}, A^{(1)}_{pq}, A^{(2)}_{pq}, \ldots, A^{(n)}_{pq}), \tag{1.3}$$

where the functional dependence of t_{ij} on the arguments may be different from that in equation (1.1).

Now, C_{pq} and $A^{(r)}_{pq}$ have the property that, if quantities $\bar{C}_{pq}$ and $\bar{A}^{(r)}_{pq}$ are defined in an analogous manner in any other rectangular Cartesian system $\bar{x}$ moving in an arbitrary manner with respect to x, then $\bar{C}_{pq}$ and C_{pq} and $\bar{A}^{(r)}_{pq}$ and $A^{(r)}_{pq}$ are the components in the systems $\bar{x}$ and x respectively of Cartesian tensors.

We now define the symmetric 3×3 matrices $\boldsymbol{T}$, $\boldsymbol{C}$ and $\boldsymbol{A}_r$ by

$$\boldsymbol{T} = \|t_{ij}\|, \quad \boldsymbol{C} = \|C_{ij}\| \quad \text{and} \quad \boldsymbol{A}_r = \|A^{(r)}_{ij}\|. \tag{1.4}$$

It has been shown [1] that, if we assume the dependence of t_{ij} on the arguments in (1.3) to be polynomial, it follows from the isotropy of the material that the stress matrix $\boldsymbol{T}$ must be expressible as a symmetric isotropic matrix polynomial in the kinematic matrices $\boldsymbol{C}$ and $\boldsymbol{A}_r$ $(r=1, 2, \ldots, n)$, the coefficients in which are polynomial scalar invariants under orthogonal transformations of the matrices $\boldsymbol{C}$ and $\boldsymbol{A}_r$ $(r=1, 2, \ldots, n)$.

If only two kinematic matrices occur in the isotropic matrix polynomial expression for $\boldsymbol{T}$, as would be the case if in (1.1) t_{ij} were taken to be a function of $\partial x_p/\partial X_q$ and $\partial v^{(1)}_p/\partial x_q$ only, or of $\partial v^{(1)}_p/\partial x_q$ and $\partial v^{(2)}_p/\partial x_q$ only, then $\boldsymbol{T}$ may be expressed in a closed form [2]. For example, if the argument matrices are $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, $\boldsymbol{T}$ may be expressed in the form

$$\begin{aligned}\boldsymbol{T} = \psi_0 \boldsymbol{I} + \psi_1 \boldsymbol{A}_1 + \psi_2 \boldsymbol{A}_2 + \psi_3 \boldsymbol{A}_1^2 + \psi_4 \boldsymbol{A}_2^2 + \psi_5 (\boldsymbol{A}_1 \boldsymbol{A}_2 + \boldsymbol{A}_2 \boldsymbol{A}_1) + \\ + \psi_6 (\boldsymbol{A}_1^2 \boldsymbol{A}_2 + \boldsymbol{A}_2 \boldsymbol{A}_1^2) + \psi_7 (\boldsymbol{A}_1 \boldsymbol{A}_2^2 + \boldsymbol{A}_2^2 \boldsymbol{A}_1) + \psi_8 (\boldsymbol{A}_1^2 \boldsymbol{A}_2^2 + \boldsymbol{A}_2^2 \boldsymbol{A}_1^2),\end{aligned} \tag{1.5}$$

where the ψ's are polynomial invariants, under orthogonal transformations of the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ and may therefore be expressed as polynomials in the integrity basis $\operatorname{tr} \boldsymbol{A}_1$, $\operatorname{tr} \boldsymbol{A}_2$, $\operatorname{tr} \boldsymbol{A}_1^2$, $\operatorname{tr} \boldsymbol{A}_2^2$, $\operatorname{tr} \boldsymbol{A}_1^3$, $\operatorname{tr} \boldsymbol{A}_2^3$, $\operatorname{tr} \boldsymbol{A}_1 \boldsymbol{A}_2$, $\operatorname{tr} \boldsymbol{A}_1 \boldsymbol{A}_2^2$, $\operatorname{tr} \boldsymbol{A}_1^2 \boldsymbol{A}_2$, $\operatorname{tr} \boldsymbol{A}_1^2 \boldsymbol{A}_2^2$.

In the present paper, we consider the more general case when the stress matrix $\boldsymbol{T}$ is an isotropic matrix polynomial in an arbitrary number of kinematic matrices $\boldsymbol{C}, \boldsymbol{A}_1, \ldots, \boldsymbol{A}_n$ and derive an expression for $\boldsymbol{T}$ as an isotropic matrix polynomial of closed form. The final result is given by Theorem 5 in § 10, by substituting $\boldsymbol{C}, \boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n$ for $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$.

In arriving at Theorem 5, the following procedure is adopted. In § 2, following a statement of certain definitions, we derive, essentially from the Hamilton-

Cayley theorem, some basic relations which are satisfied by 3×3 matrices. In § 3, these relations are used to show that any matrix polynomial in R, 3×3 matrices can be expressed as a matrix polynomial of lower or equal partial degrees in each of the matrices, of extension $\leqq R+1$ and of total degree $\leqq 2$ if $R=1$, of total degree $\leqq 5$, if $R=2$ and of total degree $\leqq R+2$, if $R>2$ (see Theorem 1). In § 4, we apply this result to determine the forms of such matrix polynomials in the cases when $R=1, 2, 3, 4$ or 5 and we also investigate the further limitations which can be made when the matrix polynomial is a symmetric matrix polynomial in symmetric matrices. In §§ 5 to 8, a finite integrity basis for R symmetric 3×3 matrices is determined. Up to this point all the results obtained are derived by algebraic operations based ultimately on the Hamilton-Cayley theorem. Finally, in §§ 9 and 10, PEANO's theorem is employed to limit further the number of elements in the integrity basis and the number of terms occurring in the closed expression for a symmetric isotropic matrix polynomial in R symmetric 3×3 matrices.

2. Some consequences of the Hamilton-Cayley theorem

We define a matrix product $\mathbf{\Pi}$ of a set of R, $n \times n$ matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ by

$$\mathbf{\Pi} = \boldsymbol{a}_{i_1}^{\alpha_1} \boldsymbol{a}_{i_2}^{\alpha_2} \ldots \boldsymbol{a}_{i_e}^{\alpha_e}, \tag{2.1}$$

where each of the subscripts $i_1, i_2, \ldots, i_e$ is one of the numbers $1, 2, \ldots, R$ and the indices $\alpha_1, \alpha_2, \ldots, \alpha_e$ are positive integers. We can, without loss of generality, write $\mathbf{\Pi}$ in the form (2.1) with no two adjacent subscripts in the sequence i_1, $i_2, \ldots, i_e$ equal and throughout this paper we shall always write matrix products in this form. Then, $\boldsymbol{a}_{i_1}^{\alpha_1}, \boldsymbol{a}_{i_2}^{\alpha_2}, \ldots, \boldsymbol{a}_{i_e}^{\alpha_e}$ are called the *factors* of the product $\mathbf{\Pi}$. These factors are said to be of degrees $\alpha_1, \alpha_2, \ldots, \alpha_e$ in the matrices $\boldsymbol{a}_{i_1}, \boldsymbol{a}_{i_2}, \ldots, \boldsymbol{a}_{i_e}$ respectively. The number of factors, e, in the matrix product $\mathbf{\Pi}$ is called its *extension*. The *total degree* of $\mathbf{\Pi}$ in the R matrices $\boldsymbol{a}_P$ is defined as $\alpha_1 + \alpha_2 + \cdots + \alpha_e$. The sum of the degrees of all the factors involving the matrix $\boldsymbol{a}_P$ is called the *partial degree* of $\mathbf{\Pi}$ in $\boldsymbol{a}_P$.

A sum of products of the type $\mathbf{\Pi}$, with scalar* coefficients, is called a *matrix polynomial* in the matrices $\boldsymbol{a}_P$. The total degree of such a polynomial is defined as the degree of its matrix product of highest total degree. The *partial degree* of the matrix polynomial in $\boldsymbol{a}_P$ is defined as that of its matrix product of highest partial degree in $\boldsymbol{a}_P$. The *extension* of the matrix polynomial is defined as the extension of its matrix product of greatest extension.

The replacement of a matrix polynomial in the matrices $\boldsymbol{a}_P$ by an identically equal matrix polynomial of lower extension will be called *contraction* of the polynomial.

If the coefficients in a matrix polynomial are invariant under all orthogonal transformations, the matrix polynomial is *isotropic***.

---

* Strictly, the term scalar is used here in the sense of a scalar matrix, rather than in its usual meaning in tensor analysis. However, if the matrix polynomial is isotropic, then the coefficients are also scalar in the latter sense.

** We might have chosen to define isotropy in the more limited sense of invariance under proper orthogonal transformations. For our present purposes, the differences which would then arise are insignificant.

In this paper we will make frequent use of a number of identities which are satisfied by 3×3 matrices, whether or not these matrices are symmetric. These identities can be derived from the fundamental relation

$$\begin{aligned}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}+\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}+\boldsymbol{c}\,\boldsymbol{a}\,\boldsymbol{b}+\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{c}+\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}+\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a}&=(\boldsymbol{b}\,\boldsymbol{c}+\boldsymbol{c}\,\boldsymbol{b})\operatorname{tr}\boldsymbol{a}+\\ &+(\boldsymbol{c}\,\boldsymbol{a}+\boldsymbol{a}\,\boldsymbol{c})\operatorname{tr}\boldsymbol{b}+(\boldsymbol{a}\,\boldsymbol{b}+\boldsymbol{b}\,\boldsymbol{a})\operatorname{tr}\boldsymbol{c}+\boldsymbol{a}(\operatorname{tr}\boldsymbol{b}\,\boldsymbol{c}-\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c})+\\ &+\boldsymbol{b}(\operatorname{tr}\boldsymbol{c}\,\boldsymbol{a}-\operatorname{tr}\boldsymbol{c}\operatorname{tr}\boldsymbol{a})+\boldsymbol{c}(\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b})+\boldsymbol{I}(\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c}-\\ &-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b}\,\boldsymbol{c}-\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c}\,\boldsymbol{a}-\operatorname{tr}\boldsymbol{c}\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}+\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}+\operatorname{tr}\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a}),\end{aligned}\tag{2.2}$$

where $\boldsymbol{I}$ denotes the 3×3 unit matrix, which is satisfied [2] by any three 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. If in (2.2) we take $\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{c}$, we obtain the Hamilton-Cayley theorem

$$\boldsymbol{a}^3-\boldsymbol{a}^2\operatorname{tr}\boldsymbol{a}+\tfrac{1}{2}\boldsymbol{a}\left[(\operatorname{tr}\boldsymbol{a})^2-\operatorname{tr}\boldsymbol{a}^2\right]-\boldsymbol{I}\det\boldsymbol{a}=0,\tag{2.3}$$

employing the relation

$$\det\boldsymbol{a}=\tfrac{1}{6}\left[(\operatorname{tr}\boldsymbol{a})^3+2\operatorname{tr}\boldsymbol{a}^3-3\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}^2\right].$$

The relation (2.2) was derived from first principles in an earlier paper [2], but it may also be derived from the Hamilton-Cayley theorem in the manner shown in the Appendix.

Taking $\boldsymbol{c}=\boldsymbol{a}$ in (2.2), we obtain

$$\begin{aligned}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}+\boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\,\boldsymbol{a}^2&=\boldsymbol{a}(\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b})+\\ &+\tfrac{1}{2}\boldsymbol{b}\left[\operatorname{tr}\boldsymbol{a}^2-(\operatorname{tr}\boldsymbol{a})^2\right]+(\boldsymbol{a}\,\boldsymbol{b}+\boldsymbol{b}\,\boldsymbol{a})\operatorname{tr}\boldsymbol{a}+\boldsymbol{a}^2\operatorname{tr}\boldsymbol{b}+\\ &+\boldsymbol{I}\left\{\operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}+\tfrac{1}{2}\operatorname{tr}\boldsymbol{b}\left[(\operatorname{tr}\boldsymbol{a})^2-\operatorname{tr}\boldsymbol{a}^2\right]\right\}.\end{aligned}\tag{2.4}$$

Multiplying equation (2.4) throughout on the right by $\boldsymbol{a}$ and on the left by $\boldsymbol{a}$, adding the equations so obtained and employing the relation (2.3) to substitute for $\boldsymbol{a}^3$, we have

$$\begin{aligned}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{b}\,\boldsymbol{a}&=\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}\operatorname{tr}\boldsymbol{a}+\boldsymbol{a}^2\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}+\\ &+\boldsymbol{a}(\operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b})-\boldsymbol{b}\det\boldsymbol{a}+\boldsymbol{I}\det\boldsymbol{a}\operatorname{tr}\boldsymbol{b}.\end{aligned}\tag{2.5}$$

Multiplying equation (2.4) throughout on the left and right by $\boldsymbol{a}$ and employing (2.3) to substitute for $\boldsymbol{a}^3$, we obtain

$$\begin{aligned}\boldsymbol{a}^2\boldsymbol{b}\,\boldsymbol{a}^2&=\tfrac{1}{2}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}\left[(\operatorname{tr}\boldsymbol{a})^2-\operatorname{tr}\boldsymbol{a}^2\right]-(\boldsymbol{a}\,\boldsymbol{b}+\boldsymbol{b}\,\boldsymbol{a})\det\boldsymbol{a}+\boldsymbol{a}^2\operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}\\ &+\boldsymbol{a}\left\{\det\boldsymbol{a}\operatorname{tr}\boldsymbol{b}-\tfrac{1}{2}\left[(\operatorname{tr}\boldsymbol{a})^2-\operatorname{tr}\boldsymbol{a}^2\right]\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\right\}+\boldsymbol{I}\det\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}.\end{aligned}\tag{2.6}$$

The results in equations (2.4), (2.5) and (2.6) have been given by Rivlin [2].

Multiplying equation (2.2) on the left and right by $\boldsymbol{a}$, we obtain

$$\begin{aligned}&\boldsymbol{a}^2\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}+\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}^2+\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}+\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{a}+\boldsymbol{a}^2\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a}+\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a}^2\\ &\quad=(\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}+\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a})\operatorname{tr}\boldsymbol{a}+(\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\,\boldsymbol{a})\operatorname{tr}\boldsymbol{b}+(\boldsymbol{a}^2\boldsymbol{b}\,\boldsymbol{a}+\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}^2)\operatorname{tr}\boldsymbol{c}+\\ &\qquad+\boldsymbol{a}^3(\operatorname{tr}\boldsymbol{b}\,\boldsymbol{c}-\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c})+\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}(\operatorname{tr}\boldsymbol{c}\,\boldsymbol{a}-\operatorname{tr}\boldsymbol{c}\operatorname{tr}\boldsymbol{a})+\\ &\qquad+\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{a}(\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b})+\boldsymbol{a}^2(\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b}\,\boldsymbol{c}-\\ &\qquad-\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c}\,\boldsymbol{a}-\operatorname{tr}\boldsymbol{c}\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}+\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}+\operatorname{tr}\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{a}).\end{aligned}\tag{2.7}$$

Employing the relation (2.3) and relations of the type (2.5) to substitute for $\boldsymbol{a}^3$, $\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{a}$, $\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2$, $\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2$ and $\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}^2$ in (2.7), we obtain the relation

$$\begin{aligned}\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}\boldsymbol{b}\boldsymbol{a} &= \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\boldsymbol{c}+\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\boldsymbol{b}+\boldsymbol{a}(\operatorname{tr}\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}\boldsymbol{c}-\operatorname{tr}\boldsymbol{a}\boldsymbol{b}\operatorname{tr}\boldsymbol{a}\boldsymbol{c})+\\ &+[\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}-\boldsymbol{b}\operatorname{tr}\boldsymbol{c}-\boldsymbol{c}\operatorname{tr}\boldsymbol{b}+\boldsymbol{I}(\operatorname{tr}\boldsymbol{b}\operatorname{tr}\boldsymbol{c}-\operatorname{tr}\boldsymbol{b}\boldsymbol{c})]\det\boldsymbol{a},\end{aligned}\tag{2.8}$$

which has been previously obtained [2] by another method. The method employed here shows that it follows as a direct consequence of equation (2.2) and hence of the Hamilton-Cayley theorem.

We define the matrix polynomial $\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})$ by

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}) &= \boldsymbol{a}^2\operatorname{tr}\boldsymbol{b}+(\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a})\operatorname{tr}\boldsymbol{a}+\boldsymbol{a}(\operatorname{tr}\boldsymbol{a}\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{b})+\\ &+\tfrac{1}{2}\boldsymbol{b}[\operatorname{tr}\boldsymbol{a}^2-(\operatorname{tr}\boldsymbol{a})^2]+\boldsymbol{I}\{\operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}-\operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\boldsymbol{b}+\tfrac{1}{2}\operatorname{tr}\boldsymbol{b}[(\operatorname{tr}\boldsymbol{a})^2-\operatorname{tr}\boldsymbol{a}^2]\}.\end{aligned}\tag{2.9}$$

From equation (2.4), we have

$$\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}+\boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2=\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}).\tag{2.10}$$

If $\boldsymbol{x}$ denotes any 3×3 matrix, we have

$$\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2=\boldsymbol{a}(\boldsymbol{a}\boldsymbol{x}\boldsymbol{b}^2).\tag{2.11}$$

Replacing $\boldsymbol{a}$ by $\boldsymbol{b}$ and $\boldsymbol{b}$ by $\boldsymbol{a}\boldsymbol{x}$ in (2.10), and employing the relation so obtained to substitute for $\boldsymbol{a}\boldsymbol{x}\boldsymbol{b}^2$ in (2.11), we obtain

$$\begin{aligned}\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2 &= -\boldsymbol{a}[\boldsymbol{b}^2\boldsymbol{a}\boldsymbol{x}+\boldsymbol{b}\boldsymbol{a}\boldsymbol{x}\boldsymbol{b}-\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{x})]\\ &= -(\boldsymbol{a}\boldsymbol{b}^2\boldsymbol{a})\boldsymbol{x}-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a})\boldsymbol{x}\boldsymbol{b}+\boldsymbol{a}\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{x}).\end{aligned}\tag{2.12}$$

We now replace $\boldsymbol{b}$ by $\boldsymbol{b}^2$ in (2.10) and use the relation so obtained to substitute for $\boldsymbol{a}\boldsymbol{b}^2\boldsymbol{a}$ in (2.12). We also use (2.10) to substitute for $\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}$ in (2.12). We then obtain

$$\begin{aligned}\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2 &= [\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2)]\boldsymbol{x}+[\boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})]\boldsymbol{x}\boldsymbol{b}+\\ &+\boldsymbol{a}\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{x})=\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{x}+\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{x}+\boldsymbol{a}^2(\boldsymbol{b}\boldsymbol{x}\boldsymbol{b})+\boldsymbol{b}(\boldsymbol{a}^2\boldsymbol{x})\boldsymbol{b}-\\ &-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2)\boldsymbol{x}-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})\boldsymbol{x}\boldsymbol{b}+\boldsymbol{a}\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{x}).\end{aligned}\tag{2.13}$$

We now replace $\boldsymbol{a}$ and $\boldsymbol{b}$ in (2.10) by $\boldsymbol{b}$ and $\boldsymbol{x}$ respectively and use the resulting relation to substitute for $\boldsymbol{b}\boldsymbol{x}\boldsymbol{b}$ in (2.13). We also replace $\boldsymbol{a}$ and $\boldsymbol{b}$ in (2.10) by $\boldsymbol{b}$ and $\boldsymbol{a}^2\boldsymbol{x}$ and use the resulting relation to substitute for $\boldsymbol{b}(\boldsymbol{a}^2\boldsymbol{x})\boldsymbol{b}$ in (2.13). We thus obtain

$$\begin{aligned}\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2 &= \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{x}+\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{x}-\boldsymbol{a}^2[\boldsymbol{b}^2\boldsymbol{x}+\boldsymbol{x}\boldsymbol{b}^2-\boldsymbol{G}(\boldsymbol{b},\boldsymbol{x})]-[\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{x}+\\ &+\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2-\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2\boldsymbol{x})]-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2)\boldsymbol{x}-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})\boldsymbol{x}\boldsymbol{b}+\boldsymbol{a}\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{x}).\end{aligned}\tag{2.14}$$

Equation (2.14) may be re-written as

$$\begin{aligned}3\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2 &= \boldsymbol{a}^2\boldsymbol{G}(\boldsymbol{b},\boldsymbol{x})-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})\boldsymbol{x}\boldsymbol{b}+\\ &+\boldsymbol{a}\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{x})-\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2)\boldsymbol{x}+\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2\boldsymbol{x}).\end{aligned}\tag{2.15}$$

Now, let $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{x}$ and $\boldsymbol{y}$ by 3×3 matrices. We have

$$\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b}^2\boldsymbol{y}\boldsymbol{c}^2=(\boldsymbol{a}^2\boldsymbol{x}\boldsymbol{b})(\boldsymbol{b}\boldsymbol{y}\boldsymbol{c}^2).\tag{2.16}$$

Replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ in (2.10) by $\boldsymbol{a}$ and $\boldsymbol{x b}$ and by $\boldsymbol{c}$ and $\boldsymbol{b y}$ and using the relations so obtained to substitute for $\boldsymbol{a^2 x b}$ and $\boldsymbol{b y c^2}$ in (2.16), we have

$$\begin{aligned}\boldsymbol{a^2 x\, b^2 y\, c^2} &= [\boldsymbol{a x b a} + \boldsymbol{x b a^2} - G(\boldsymbol{a}, \boldsymbol{x b})]\,[\boldsymbol{c^2 b y} + \boldsymbol{c b y c} - G(\boldsymbol{c}, \boldsymbol{b y})] \\ &= \boldsymbol{a x}(\boldsymbol{b a c^2 b})\,\boldsymbol{y} + \boldsymbol{a x}(\boldsymbol{b a c b})\,\boldsymbol{y c} + \boldsymbol{x}(\boldsymbol{b a^2 c^2 b})\,\boldsymbol{y} + \\ &\quad + \boldsymbol{x}(\boldsymbol{b a^2 c b})\,\boldsymbol{y c} - (\boldsymbol{a x b a} + \boldsymbol{x b a^2})\, G(\boldsymbol{c}, \boldsymbol{b y}) - \\ &\quad - G(\boldsymbol{a}, \boldsymbol{x b})\,(\boldsymbol{c^2 b y} + \boldsymbol{c b y c}) + G(\boldsymbol{a}, \boldsymbol{x b})\, G(\boldsymbol{c}, \boldsymbol{b y}).\end{aligned}$$

Employing relations of the type (2.10) to substitute for $\boldsymbol{b a c^2 b}$, $\boldsymbol{b a c b}$, $\boldsymbol{b a^2 c^2 b}$, $\boldsymbol{b a^2 c b}$, $\boldsymbol{a x b a} + \boldsymbol{x b a^2}$ and $\boldsymbol{c^2 b y} + \boldsymbol{c b y c}$, we obtain

$$\begin{aligned}\boldsymbol{a^2 x\, b^2 y\, c^2} &= -\boldsymbol{a x}[\boldsymbol{b^2 a c^2} + \boldsymbol{a c^2 b^2} - G(\boldsymbol{b}, \boldsymbol{a c^2})]\,\boldsymbol{y} - \\ &\quad - \boldsymbol{a x}[\boldsymbol{b^2 a c} + \boldsymbol{a c b^2} - G(\boldsymbol{b}, \boldsymbol{a c})]\,\boldsymbol{y c} - \\ &\quad - \boldsymbol{x}[\boldsymbol{b^2 a^2 c^2} + \boldsymbol{a^2 c^2 b^2} - G(\boldsymbol{b}, \boldsymbol{a^2 c^2})]\,\boldsymbol{y} - \\ &\quad - \boldsymbol{x}[\boldsymbol{b^2 a^2 c} + \boldsymbol{a^2 c b^2} - G(\boldsymbol{b}, \boldsymbol{a^2 c})]\,\boldsymbol{y c} + \\ &\quad + [\boldsymbol{a^2 x b} - G(\boldsymbol{a}, \boldsymbol{x b})]\, G(\boldsymbol{c}, \boldsymbol{b y}) + \\ &\quad + G(\boldsymbol{a}, \boldsymbol{x b})\,[\boldsymbol{b y c^2} - G(\boldsymbol{c}, \boldsymbol{b y})] + G(\boldsymbol{a}, \boldsymbol{x b})\, G(\boldsymbol{c}, \boldsymbol{b y}) \\ &= -(\boldsymbol{a x b^2 a} + \boldsymbol{x b^2 a^2})\,(\boldsymbol{c^2 y} + \boldsymbol{c y c}) - \\ &\quad - (\boldsymbol{a x a} + \boldsymbol{x a^2})\,(\boldsymbol{c^2 b^2 y} + \boldsymbol{c b^2 y c}) + \boldsymbol{a^2 x b}\, G(\boldsymbol{c}, \boldsymbol{b y}) + \\ &\quad + G(\boldsymbol{a}, \boldsymbol{x b})\,\boldsymbol{b y c^2} + \boldsymbol{a x}\, G(\boldsymbol{b}, \boldsymbol{a c})\,\boldsymbol{y c} + \boldsymbol{a x}\, G(\boldsymbol{b}, \boldsymbol{a c^2})\,\boldsymbol{y} + \\ &\quad + \boldsymbol{x}\, G(\boldsymbol{b}, \boldsymbol{a^2 c})\,\boldsymbol{y c} + \boldsymbol{x}\, G(\boldsymbol{b}, \boldsymbol{a^2 c^2})\,\boldsymbol{y} - G(\boldsymbol{a}, \boldsymbol{x b})\, G(\boldsymbol{c}, \boldsymbol{b y}).\end{aligned} \tag{2.17}$$

Again employing relations of the type (2.10) to substitute for $\boldsymbol{a x b^2 a} + \boldsymbol{x b^2 a^2}$, $\boldsymbol{c^2 y} + \boldsymbol{c y c}$, $\boldsymbol{a x a} + \boldsymbol{x a^2}$ and $\boldsymbol{c^2 b^2 y} + \boldsymbol{c b^2 y c}$ in (2.17), we obtain

$$\begin{aligned}\boldsymbol{a^2 x\, b^2 y\, c^2} &= -[\boldsymbol{a^2 x b^2} - G(\boldsymbol{a}, \boldsymbol{x b^2})]\,[\boldsymbol{y c^2} - G(\boldsymbol{c}, \boldsymbol{y})] - \\ &\quad - [\boldsymbol{a^2 x} - G(\boldsymbol{a}, \boldsymbol{x})]\,[\boldsymbol{b^2 y c^2} - G(\boldsymbol{c}, \boldsymbol{b^2 y})] + \\ &\quad + \boldsymbol{a^2 x b}\, G(\boldsymbol{c}, \boldsymbol{b y}) + G(\boldsymbol{a}, \boldsymbol{x b})\,\boldsymbol{b y c^2} + \\ &\quad + \boldsymbol{a x}\, G(\boldsymbol{b}, \boldsymbol{a c})\,\boldsymbol{y c} + \boldsymbol{a x}\, G(\boldsymbol{b}, \boldsymbol{a c^2})\,\boldsymbol{y} + \\ &\quad + \boldsymbol{x}\, G(\boldsymbol{b}, \boldsymbol{a^2 c})\,\boldsymbol{y c} + \boldsymbol{x}\, G(\boldsymbol{b}, \boldsymbol{a^2 c^2})\,\boldsymbol{y} - G(\boldsymbol{a}, \boldsymbol{x b})\, G(\boldsymbol{c}, \boldsymbol{b y}).\end{aligned} \tag{2.18}$$

Whence,

$$\begin{aligned}3\boldsymbol{a^2 x\, b^2 y\, c^2} &= \boldsymbol{a^2 x b^2}\, G(\boldsymbol{c}, \boldsymbol{y}) + G(\boldsymbol{a}, \boldsymbol{x})\,\boldsymbol{b^2 y c^2} + \boldsymbol{a^2 x b}\, G(\boldsymbol{c}, \boldsymbol{b y}) + \\ &\quad + G(\boldsymbol{a}, \boldsymbol{x b})\,\boldsymbol{b y c^2} + \boldsymbol{a x}\, G(\boldsymbol{b}, \boldsymbol{a c})\,\boldsymbol{y c} + \boldsymbol{a^2 x}\, G(\boldsymbol{c}, \boldsymbol{b^2 y}) + \\ &\quad + G(\boldsymbol{a}, \boldsymbol{x b^2})\,\boldsymbol{y c^2} + \boldsymbol{a x}\, G(\boldsymbol{b}, \boldsymbol{a c^2})\,\boldsymbol{y} + \boldsymbol{x}\, G(\boldsymbol{b}, \boldsymbol{a^2 c})\,\boldsymbol{y c} + \\ &\quad + \boldsymbol{x}\, G(\boldsymbol{b}, \boldsymbol{a^2 c^2})\,\boldsymbol{y} - G(\boldsymbol{a}, \boldsymbol{x b^2})\, G(\boldsymbol{c}, \boldsymbol{y}) - \\ &\quad - G(\boldsymbol{a}, \boldsymbol{x})\, G(\boldsymbol{c}, \boldsymbol{b^2 y}) - G(\boldsymbol{a}, \boldsymbol{x b})\, G(\boldsymbol{c}, \boldsymbol{b y}).\end{aligned} \tag{2.19}$$

We note that a different expression for $3\boldsymbol{a^2 x b^2 y c^2}$, when $\boldsymbol{x} \neq \boldsymbol{I}$, is obtained from (2.15) by multiplying the equation throughout on the right by $\boldsymbol{y c^2}$.

Taking $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{I}$ in (2.19) and noting, from (2.9), that

$$G(\boldsymbol{c}, \boldsymbol{I}) = 3\boldsymbol{c}^2, \qquad G(\boldsymbol{a}, \boldsymbol{I}) = 3\boldsymbol{a}^2,$$

we obtain

$$\begin{aligned}3\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}^2 = &-\boldsymbol{a}^2\boldsymbol{b}\,\boldsymbol{G}(\boldsymbol{c},\boldsymbol{b}) - \boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})\,\boldsymbol{b}\,\boldsymbol{c}^2 + 2\boldsymbol{a}^2\boldsymbol{G}(\boldsymbol{c},\boldsymbol{b}^2) + \\ &+ 2\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2)\,\boldsymbol{c}^2 - \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{c})\,\boldsymbol{c} - \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}\boldsymbol{c}^2) - \\ &- \boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2\boldsymbol{c})\,\boldsymbol{c} - \boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2\boldsymbol{c}^2) + \boldsymbol{G}(\boldsymbol{a},\boldsymbol{b})\,\boldsymbol{G}(\boldsymbol{c},\boldsymbol{b})\,.\end{aligned} \tag{2.20}$$

We now consider the product $\boldsymbol{a}\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{y}$ are 3×3 matrices. We have

$$\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} = \boldsymbol{a}\,\boldsymbol{b}^2(\boldsymbol{a}^2\boldsymbol{y})\,. \tag{2.21}$$

Replacing $\boldsymbol{b}$ in (2.10) by $\boldsymbol{y}$ and using the resulting relation to substitute for $\boldsymbol{a}^2\boldsymbol{y}$ in (2.21), we obtain

$$\begin{aligned}\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} &= \boldsymbol{a}\,\boldsymbol{b}^2\left[-\boldsymbol{a}\,\boldsymbol{y}\,\boldsymbol{a} - \boldsymbol{y}\,\boldsymbol{a}^2 + \boldsymbol{G}(\boldsymbol{a},\boldsymbol{y})\right] \\ &= -\boldsymbol{a}(\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y})\,\boldsymbol{a} - \boldsymbol{a}(\boldsymbol{b}^2\boldsymbol{y}\,\boldsymbol{a}^2) + \boldsymbol{a}\,\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{y})\,.\end{aligned} \tag{2.22}$$

Again, using a relation of the type (2.10) to substitute for $\boldsymbol{a}(\boldsymbol{b}^2\boldsymbol{a}\boldsymbol{y})\boldsymbol{a}$ in (2.22), we obtain

$$\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} = \boldsymbol{a}^2(\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y}) + (\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y})\,\boldsymbol{a}^2 - \boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y}) - \boldsymbol{a}(\boldsymbol{b}^2\boldsymbol{y}\,\boldsymbol{a}^2) + \boldsymbol{a}\,\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{y})\,,$$

i.e.

$$\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} - \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y} = \boldsymbol{b}^2(\boldsymbol{a}\,\boldsymbol{y})\,\boldsymbol{a}^2 - \boldsymbol{a}(\boldsymbol{b}^2\boldsymbol{y})\,\boldsymbol{a}^2 - \boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y}) + \boldsymbol{a}\,\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{y})\,. \tag{2.23}$$

Now, in (2.15) we replace $\boldsymbol{a}$, $\boldsymbol{x}$ and $\boldsymbol{b}$ by $\boldsymbol{b}$, $\boldsymbol{a}\boldsymbol{y}$ and $\boldsymbol{a}$ respectively and use the relation so obtained to substitute for $\boldsymbol{b}^2(\boldsymbol{a}\boldsymbol{y})\boldsymbol{a}^2$ in (2.23). Also, in (2.15) we replace $\boldsymbol{a}$, $\boldsymbol{x}$ and $\boldsymbol{b}$ by $\boldsymbol{b}$, $\boldsymbol{y}$ and $\boldsymbol{a}$ respectively, and use the relation so obtained to substitute for $\boldsymbol{b}^2\boldsymbol{y}\boldsymbol{a}^2$ in (2.23). We thus obtain

$$\begin{aligned}\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} - \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y} = \tfrac{1}{3}\big[&\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{a}\,\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{b},\boldsymbol{a})\,\boldsymbol{a}\,\boldsymbol{y}\,\boldsymbol{a} + \\ &+ \boldsymbol{b}\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2)\,\boldsymbol{a}\,\boldsymbol{y} + \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a})\,\boldsymbol{y}\,\boldsymbol{a} - \\ &- \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}\,\boldsymbol{y}) + \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2)\,\boldsymbol{y} - \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2\boldsymbol{y}) - \\ &- 2\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y}) + 2\boldsymbol{a}\,\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{y})\big]\,.\end{aligned} \tag{2.24}$$

With the notation

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{a},\boldsymbol{b}) = &\,\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}\operatorname{tr}\boldsymbol{a} + \boldsymbol{a}^2\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b} + \\ &+ \boldsymbol{a}(\operatorname{tr}\boldsymbol{a}^2\boldsymbol{b} - \operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}) - \boldsymbol{b}\det\boldsymbol{a} + \boldsymbol{I}\det\boldsymbol{a}\operatorname{tr}\boldsymbol{b}\,,\end{aligned} \tag{2.25}$$

equation (2.5) may be re-written as

$$\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{a}^2 + \boldsymbol{a}^2\boldsymbol{b}\,\boldsymbol{a} = \boldsymbol{H}(\boldsymbol{a},\boldsymbol{b})\,. \tag{2.26}$$

Replacing $\boldsymbol{b}$ in (2.26) by $\boldsymbol{b}^2$, and multiplying the relation so obtained on the right by $\boldsymbol{y}$, we have

$$\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} + \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{a},\boldsymbol{b}^2)\,\boldsymbol{y}\,. \tag{2.27}$$

From equations (2.24) and (2.27), we obtain

$$\begin{aligned}2\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{y} = \boldsymbol{H}(\boldsymbol{a},\boldsymbol{b}^2)\,\boldsymbol{y} + \tfrac{1}{3}\big[&\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{a}\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{b},\boldsymbol{a})\,\boldsymbol{a}\,\boldsymbol{y}\,\boldsymbol{a} + \\ &+ \boldsymbol{b}\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{y}) - \boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2)\,\boldsymbol{a}\,\boldsymbol{y} + \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a})\,\boldsymbol{y}\,\boldsymbol{a} - \\ &- \boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}\,\boldsymbol{y}) + \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{b},\boldsymbol{a}^2)\,\boldsymbol{y} - \boldsymbol{a}\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2\boldsymbol{y}) - \\ &- 2\boldsymbol{G}(\boldsymbol{a},\boldsymbol{b}^2\boldsymbol{a}\,\boldsymbol{y}) + 2\boldsymbol{a}\,\boldsymbol{b}^2\,\boldsymbol{G}(\boldsymbol{a},\boldsymbol{y})\big]\end{aligned} \tag{2.28}$$

and
$$\begin{aligned} 2a^2 b^2 a y = H(a, b^2)\, y - \tfrac{1}{3}\big[& b^2 G(a, a y) - G(b, a)\, a y a + \\ & + b\, G(a, b a y) - G(b, a^2)\, a y + a\, G(b, a)\, y a - \\ & - a b\, G(a, b y) + a\, G(b, a^2)\, y - a\, G(a, b^2 y) - \\ & - 2G(a, b^2 a y) + 2a b^2 G(a, y)\big]. \end{aligned} \tag{2.29}$$

Analogous expressions may be derived in a somewhat similar manner for the matrix products $y a b^2 a^2$ and $y a^2 b^2 a$. Alternatively, such expressions may be derived from equations (2.28) and (2.29) by equating the transposes of both sides of the equations. We then make use of the fact that the transpose of a product of matrices is equal to the product of the transposes of these matrices with the order of the factors reversed. Finally, replacing the transposes of the matrices a, b and y by a, b and y respectively, the desired expressions for $y a^2 b^2 a$ and $y a b^2 a^2$ are obtained.

Provided that $x \neq I$ and $y \neq I$, equations (2.15), (2.19), (2.20), (2.28) and (2.29) express the matrix products $a^2 x b^2$, $a^2 x b^2 y c^2$, $a^2 b^2 c^2$, $a b^2 a^2 y$ and $a^2 b^2 a y$ respectively as matrix polynomials of lower total degree.

3. The contraction of matrix polynomials in R, 3×3 matrices

The contraction of matrix polynomials in R, 2×2 matrices and in two, 3×3 matrices has been treated by RIVLIN [2]. In this section, we discuss the contraction of matrix polynomials in R, 3×3 matrices a_P ($P = 1, 2, \dots, R$). The particular cases when $R = 3$, 4 and 5 will be discussed in greater detail in § 4.

Any matrix polynomial P in R, 3×3 matrices is the sum of matrix products of the type Π, given by equation (2.1), with scalar coefficients. If any of the indices $\alpha_1, \alpha_2, \dots, \alpha_e$ occurring in the product Π is greater than two, then we can use the Hamilton-Cayley theorem to express Π as a matrix polynomial in which none of the matrix products has a factor of degree greater than two. In this way we readily obtain

Lemma 1. *Any matrix product* Π *of the* R, 3×3 *matrices* a_P ($P = 1, 2, \dots, R$) *can be expressed as a matrix polynomial, with extension equal to that of* Π, *in which each of the matrix products is either the unit matrix* I *or is formed from some or all of the factors* a_P, a_P^2 ($P = 1, 2, \dots, R$) *and has partial degree in* a_P *less than or equal to that of* Π.

Each matrix product in the matrix polynomial, which is not I, therefore takes the form (2.1) with each of the α's either 1 or 2. If any factor in such a matrix product occurs twice, then a relation of the type (2.4) can be used to express the matrix product as a matrix polynomial of lower extension. Let us consider a matrix product in the matrix polynomial of the form

$$y\, a_K^2\, x\, a_K^2\, z \quad \text{or} \quad y\, a_K\, x\, a_K\, z, \tag{3.1}$$

where y and z are either matrix products or the unit matrix I and x is a matrix product. Then, replacing a and b in (2.4) by a_K^2 and x respectively or by a_K and x respectively, we can express either of the products (3.1) as a matrix polynomial of lower extension. Using Lemma 1, we see that this may in turn be

expressed as a matrix polynomial in which each matrix product is formed from some or all of the factors $\boldsymbol{a}_P$ and $\boldsymbol{a}_P^2$. By repeated application of this procedure we obtain

Lemma 2. *Any matrix polynomial* $\boldsymbol{P}$ *in the* R, 3×3 *matrices* $\boldsymbol{a}_P$ *may be expressed as a matrix polynomial* $\boldsymbol{Q}$ *of lower or equal extension and total degree, each matrix product in which has partial degrees in each of the matrices* $\boldsymbol{a}_P$ *lower than or equal to those of* $\boldsymbol{P}$, *is either* $\boldsymbol{I}$ *or is formed from some or all of the factors* $\boldsymbol{a}_P, \boldsymbol{a}_P^2$ $(P=1, 2, \ldots, R)$ *and does not contain two identical factors.*

Suppose a matrix product in the polynomial $\boldsymbol{Q}$ has the form

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{x}\,\boldsymbol{a}_L^2\,\boldsymbol{z} \qquad (K \neq L), \tag{3.2}$$

where $\boldsymbol{y}\boldsymbol{x}\boldsymbol{z}$ is a matrix product formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ and $\boldsymbol{a}_Q^2$ $(Q=1, 2, \ldots, R;\ Q \neq K, L)$, no factor occurring twice in the product, and $\boldsymbol{x} \neq \boldsymbol{I}$. Replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ in (2.15) by $\boldsymbol{a}_K$ and $\boldsymbol{a}_L$ and using the relation so obtained to substitute for $\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_L^2$ in (3.2), we see that the product (3.2) can be expressed as a matrix polynomial of lower total degree in the matrices $\boldsymbol{a}_P$ than (3.2). Each matrix product in this polynomial is of lower or equal partial degree in each of the matrices $\boldsymbol{a}_P$ than (3.2). By repeated application of this procedure and employing lemma 2 we obtain

Lemma 3. *Any matrix polynomial* $\boldsymbol{P}$ *in* R, 3×3 *matrices* $\boldsymbol{a}_P$ *can be expressed as a matrix polynomial* $\boldsymbol{R}$ *of lower or equal total degree, in which each matrix product is of lower or equal partial degree in each of the matrices* $\boldsymbol{a}_P$ *than* $\boldsymbol{P}$, *is either* $\boldsymbol{I}$ *or is formed from some or all of the factors* $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ *and at most two of the factors* $\boldsymbol{a}_Q^2$ $(Q=1, 2, \ldots, R)$, *no matrix product contains two identical factors and any matrix product containing two factors* $\boldsymbol{a}_Q^2$ *contains them consecutively.*

We obtain immediately the

Corollary. *Any matrix polynomial* $\boldsymbol{P}$ *in* R, 3×3 *matrices* $\boldsymbol{a}_P$ *can be expressed as a matrix polynomial of total degree* $\leqq R+4$ *and of extension* $\leqq R+2$.

As a result of Lemma 3, we see that any matrix polynomial $\boldsymbol{P}$ in the R matrices $\boldsymbol{a}_P$ can be expressed as a matrix polynomial $\boldsymbol{R}$, in which each matrix product has one or other of the forms

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{z} \qquad (K \neq L), \tag{3.3}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{z} \tag{3.4}$$

or

$$\boldsymbol{y}\,\boldsymbol{z}, \tag{3.5}$$

where $\boldsymbol{y}$ and $\boldsymbol{z}$ are either $\boldsymbol{I}$ or matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ such that no two factors in $\boldsymbol{y}\boldsymbol{z}$ are the same.

We consider first a product of the form (3.3). Four possibilities arise which may be considered separately:

(i) $\boldsymbol{a}_K$ is a factor of $\boldsymbol{y}\boldsymbol{z}$, but $\boldsymbol{a}_L$ is not a factor of $\boldsymbol{y}\boldsymbol{z}$,

(ii) $\boldsymbol{a}_L$ is a factor of $\boldsymbol{y}\boldsymbol{z}$, but $\boldsymbol{a}_K$ is not a factor of $\boldsymbol{y}\boldsymbol{z}$,

(iii) $\boldsymbol{a}_K$ and $\boldsymbol{a}_L$ are both factors of $\boldsymbol{y}\boldsymbol{z}$,

(iv) neither $\boldsymbol{a}_K$ nor $\boldsymbol{a}_L$ is a factor of $\boldsymbol{y}\boldsymbol{z}$.

In case *(i)* we may write the matrix product (3.3) in one of the forms

$$\boldsymbol{y}\,\boldsymbol{a}_K\,\boldsymbol{v}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{z}, \tag{3.6}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{v}\,\boldsymbol{a}_K\,\boldsymbol{z}, \tag{3.7}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{a}_K\,\boldsymbol{z}, \tag{3.8}$$

where $\boldsymbol{y}$, $\boldsymbol{v}$ and $\boldsymbol{z}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$; $P \neq K, L$) such that, in (3.6) and (3.7), no two factors in $\boldsymbol{yvz}$ are the same and, in (3.8), no two factors in $\boldsymbol{yz}$ are the same. As special cases, $\boldsymbol{y}$ or $\boldsymbol{z}$ or both may be the unit matrix $\boldsymbol{I}$, but $\boldsymbol{v} \neq \boldsymbol{I}$. If the matrix product has the form (3.6), then taking $\boldsymbol{a}=\boldsymbol{a}_K$ and $\boldsymbol{b}=\boldsymbol{v}$ in equation (2.5), and multiplying the resulting equation throughout on the left and right by $\boldsymbol{y}$ and $\boldsymbol{a}_L^2\boldsymbol{z}$ respectively, we obtain

$$\boldsymbol{y}\,\boldsymbol{a}_K\,\boldsymbol{v}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{z} = -\,\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{v}\,\boldsymbol{a}_K\,\boldsymbol{a}_L^2\,\boldsymbol{z} + \boldsymbol{P}_1,$$

where $\boldsymbol{P}_1$ is a matrix polynomial in which each of the matrix products is of lower total degree than (3.6) and of partial degree in each $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) less than or equal to that of (3.6). Taking $\boldsymbol{a}=\boldsymbol{a}_K$, $\boldsymbol{x}=\boldsymbol{v}\boldsymbol{a}_K$, $\boldsymbol{b}=\boldsymbol{a}_L$ in (2.15), we can express $\boldsymbol{y}\boldsymbol{a}_K^2\boldsymbol{v}\boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{z}$ and hence the matrix product (3.6) as a matrix polynomial in which each of the matrix products is of lower total degree than (3.6) and of partial degree in each $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) less than or equal to that of (3.6). By an analogous procedure, taking $\boldsymbol{a}=\boldsymbol{a}_K$ and $\boldsymbol{b}=\boldsymbol{a}_L^2\boldsymbol{v}$ in (2.5) and $\boldsymbol{a}=\boldsymbol{a}_L$, $\boldsymbol{x}=\boldsymbol{v}$, $\boldsymbol{b}=\boldsymbol{a}_K$ in (2.15), we see that (3.7) may be expressed as a matrix polynomial in which each of the matrix products is of lower total degree than (3.7) and of partial degree in each $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) less than or equal to that of (3.7). If the matrix product has the form (3.8), we take $\boldsymbol{a}=\boldsymbol{a}_K$, $\boldsymbol{b}=\boldsymbol{a}_L$ and $\boldsymbol{y}=\boldsymbol{z}$ in (2.29) and use the resulting relation to express (3.8) as a matrix polynomial in which each of the matrix products is of lower total degree and of lower or equal partial degree in each $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) than (3.8), provided that $\boldsymbol{z} \neq \boldsymbol{I}$. When $\boldsymbol{z}=\boldsymbol{I}$, (3.8) takes the form $\boldsymbol{y}\boldsymbol{a}_K^2\boldsymbol{a}_L^2\boldsymbol{a}_K$, which may be expressed as a matrix polynomial in which each of the terms is of lower total degree and lower or equal partial degree in each $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), by means of a relation analogous to (2.28), unless $\boldsymbol{y}=\boldsymbol{I}$, in which case (3.8) becomes $\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{a}_K$. Thus, any matrix product of the form (3.3), which satisfies the conditions of case *(i)*, except $\boldsymbol{a}_K^2\boldsymbol{a}_L^2\boldsymbol{a}_K$, can be expressed as a matrix polynomial in which each of the matrix products is of lower total degree and of lower or equal partial degree in each of the matrices $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$).

In case *(ii)* we may write the matrix product (3.3) in one of the forms

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{v}\,\boldsymbol{a}_L\,\boldsymbol{z}, \tag{3.9}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_L\,\boldsymbol{v}\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{z}, \tag{3.10}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_L\,\boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{z}, \tag{3.11}$$

where in (3.9) and (3.10) $\boldsymbol{y}$, $\boldsymbol{v}$ and $\boldsymbol{z}$ have the same meanings as in (3.6) and (3.7), and in (3.11) $\boldsymbol{y}$ and $\boldsymbol{z}$ have the same meanings as in (3.8). In a manner analogous to that employed in discussing the matrix products (3.6), (3.7) and (3.8), we can show that each of the matrix products (3.9), (3.10) and (3.11),

with the exception of $\boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{a}_L^2$, can be expressed as a matrix polynomial in which each of the matrix products is of lower total degree and of lower or equal partial degree in each of the matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$. We note also, from (2.26), that

$$\boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{a}_K = -\boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 + \boldsymbol{H}(\boldsymbol{a}_K, \boldsymbol{a}_L^2),$$

where $\boldsymbol{H}(\boldsymbol{a}, \boldsymbol{b})$ is defined as in (2.25). $\boldsymbol{H}(\boldsymbol{a}, \boldsymbol{b})$ is, of course, a matrix polynomial in which each of the terms is of lower total degree and of lower or equal partial degree in $\boldsymbol{a}$ and $\boldsymbol{b}$ than $\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2$ and $\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a}$.

In case *(iii)*, we may write the matrix product (3.3) in one of the forms

$$\boldsymbol{y}\, \boldsymbol{a}_K \boldsymbol{v}\, \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{u}\, \boldsymbol{a}_L \boldsymbol{z}, \tag{3.12}$$

or

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{v}\, \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{u}\, \boldsymbol{a}_K \boldsymbol{z}, \tag{3.13}$$

or

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{v}\, \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{z}, \tag{3.14}$$

or

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{v}\, \boldsymbol{a}_K \boldsymbol{z}, \tag{3.15}$$

or

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{z}. \tag{3.16}$$

In these matrix products $\boldsymbol{y}, \boldsymbol{v}, \boldsymbol{u}, \boldsymbol{z}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R;\ P \neq K, L)$ and $\boldsymbol{y}$ and $\boldsymbol{z}$ may be the unit matrix $\boldsymbol{I}$, while $\boldsymbol{u} \neq \boldsymbol{I}$ and $\boldsymbol{v} \neq \boldsymbol{I}$. In (3.12) and (3.13) no two factors of $\boldsymbol{y}\boldsymbol{v}\boldsymbol{u}\boldsymbol{z}$ are the same; in (3.14) and (3.15) no two factors of $\boldsymbol{y}\boldsymbol{v}\boldsymbol{z}$ are the same and in (3.16) no two factors of $\boldsymbol{y}\boldsymbol{z}$ are the same. Each of the matrix products (3.12) to (3.16) may be expressed as a matrix polynomial in which each matrix product is of lower total degree and of lower or equal partial degree in each of the matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$. For each of the matrix products (3.12) to (3.15), this can be shown by employing equations (2.5) and (2.15) in a manner similar to that adopted in discussing the matrix products (3.6), (3.7), (3.9) and (3.10). Thus, for the matrix product (3.12) we take $\boldsymbol{a}=\boldsymbol{a}_K$, $\boldsymbol{b}=\boldsymbol{v}$ in equation (2.5) and $\boldsymbol{a}=\boldsymbol{a}_K$, $\boldsymbol{x}=\boldsymbol{v}\boldsymbol{a}_K$, $\boldsymbol{b}=\boldsymbol{a}_L$ in equation (2.15). For (3.13) and (3.14), we take $\boldsymbol{a}=\boldsymbol{a}_L$, $\boldsymbol{b}=\boldsymbol{v}\boldsymbol{a}_K^2$ in (2.5) and $\boldsymbol{a}=\boldsymbol{a}_L$, $\boldsymbol{x}=\boldsymbol{v}$, $\boldsymbol{b}=\boldsymbol{a}_K$ in (2.15). For (3.15) we take $\boldsymbol{a}=\boldsymbol{a}_K$, $\boldsymbol{b}=\boldsymbol{a}_L^2\boldsymbol{v}$ in (2.5) and $\boldsymbol{a}=\boldsymbol{a}_L$, $\boldsymbol{x}=\boldsymbol{v}$, $\boldsymbol{b}=\boldsymbol{a}_K$ in (2.15). We now consider the matrix product (3.16). We take $\boldsymbol{a}=\boldsymbol{a}_K$ and $\boldsymbol{b}=\boldsymbol{a}_L^2$ in (2.5) and multiply the resulting equation throughout on the left and right by $\boldsymbol{y}\boldsymbol{a}_L$ and $\boldsymbol{z}$ respectively, to obtain

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{z} = -\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} + \boldsymbol{P}_2 \tag{3.17}$$

where $\boldsymbol{P}_2$ is a matrix polynomial in which each of the matrix products is of lower total degree than (3.16) and of partial degree in each $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ less than or equal to that of (3.16). Again taking $\boldsymbol{a}=\boldsymbol{a}_L$ and $\boldsymbol{b}=\boldsymbol{a}_K$ in (2.5) and multiplying the equation so obtained throughout on the left and right by $\boldsymbol{y}$ and $\boldsymbol{a}_K^2\boldsymbol{z}$ respectively, we obtain

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} = -\boldsymbol{y}\, \boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{z} + \boldsymbol{P}_3 \tag{3.18}$$

where $\boldsymbol{P}_3$ is a matrix polynomial in which each of the matrix products is of lower total degree than (3.16) and of partial degree in each $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ less than or equal to that of (3.16). Combining (3.17) and (3.18), we have

$$\boldsymbol{y}\, \boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{z} = \boldsymbol{y}\, \boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{z} + \boldsymbol{P}_2 - \boldsymbol{P}_3. \tag{3.19}$$

Taking $\boldsymbol{a}=\boldsymbol{a}_L$, $\boldsymbol{x}=\boldsymbol{a}_K\boldsymbol{a}_L$, $\boldsymbol{b}=\boldsymbol{a}_K$ in (2.15), we see that the matrix product (3.16) can be expressed as a matrix polynomial in which each of the matrix products is of lower total degree than (3.16) and of partial degree in each $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ less than or equal to that of (3.16).

We now consider a matrix product of the form (3.4). This may be written in one or other of the forms

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{v}\,\boldsymbol{a}_K\,\boldsymbol{z}, \tag{3.20}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_K\,\boldsymbol{v}\,\boldsymbol{a}_K^2\,\boldsymbol{z}, \tag{3.21}$$

or

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{z}, \tag{3.22}$$

where $\boldsymbol{y}, \boldsymbol{v}$ and $\boldsymbol{z}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R;\ P\neq K)$ such that, in (3.20) and (3.21), no two factors of $\boldsymbol{y}\boldsymbol{v}\boldsymbol{z}$ are the same and, in (3.22), no two factors of $\boldsymbol{y}\boldsymbol{z}$ are the same. As special cases, $\boldsymbol{y}$ and $\boldsymbol{z}$ may be the unit matrix $\boldsymbol{I}$, but $\boldsymbol{v}\neq\boldsymbol{I}$. We note that by taking $\boldsymbol{a}=\boldsymbol{a}_K$ and $\boldsymbol{b}=\boldsymbol{v}$ in (2.5) and multiplying the resulting equation throughout on the left and right by $\boldsymbol{y}$ and $\boldsymbol{z}$ respectively, we obtain

$$\boldsymbol{y}\,\boldsymbol{a}_K^2\,\boldsymbol{v}\,\boldsymbol{a}_K\,\boldsymbol{z} = -\,\boldsymbol{y}\,\boldsymbol{a}_K\,\boldsymbol{v}\,\boldsymbol{a}_K^2\,\boldsymbol{z} + \boldsymbol{P}_4$$

where $\boldsymbol{P}_4$ is a matrix polynomial, in which each of the matrix products is of lower total degree than (3.20) and of partial degree in each $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ less than or equal to that of (3.20).

Combining the foregoing results, we readily obtain

Theorem 1. *Any matrix polynomial $\boldsymbol{P}$ in R, 3×3 matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ can be expressed as a matrix polynomial of lower or equal extension and total degree, in which*

(i) each matrix product is either $\boldsymbol{I}$ or is formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ and at most two of the factors $\boldsymbol{a}_P^2$ $(P=1, 2, \ldots, R)$;

(ii) no two factors in a single matrix product are the same;

(iii) each matrix product is of lower or equal partial degree in each of the matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ than the matrix polynomial $\boldsymbol{P}$;

(iv) matrix products containing two of the factors $\boldsymbol{a}_P^2$ $(P=1, 2, \ldots, R)$ contain them consecutively;

(v) no matrix product containing both of the factors $\boldsymbol{a}_K^2$ and $\boldsymbol{a}_L^2$ contains either of the factors $\boldsymbol{a}_K$ and $\boldsymbol{a}_L$, unless it has the form $\boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_K^2$;

(vi) $\boldsymbol{a}_K$ precedes $\boldsymbol{a}_K^2$ in any matrix product containing both $\boldsymbol{a}_K$ and $\boldsymbol{a}_L^2$ as factors.

From the conditions (i), (ii) and (v) in Theorem 1, we obtain immediately the

Corollary 1. *Any matrix polynomial in R, 3×3 matrices $\boldsymbol{a}_P$ can be expressed as a matrix polynomial of extension $\leq R+1$ and of total degree ≤ 2 if $R=1$, of total degree ≤ 5 if $R=2$ and of total degree $\leq R+2$ if $R>2$.*

If the matrix polynomial $\boldsymbol{P}$ is symmetric, then

$$\boldsymbol{P} = \tfrac{1}{2}(\boldsymbol{P}+\boldsymbol{P}'),$$

where $\boldsymbol{P}'$ is the transpose of $\boldsymbol{P}$. Then if $\boldsymbol{P}$ contains the term $\boldsymbol{a}_K\boldsymbol{a}_L^2\boldsymbol{a}_K^2$ with a scalar coefficient, this term may be replaced by the term

$$\tfrac{1}{2}\,(\boldsymbol{a}_K\,\boldsymbol{a}_L^2\,\boldsymbol{a}_K^2 + \boldsymbol{a}_K^2\,\boldsymbol{a}_L^2\,\boldsymbol{a}_K) \tag{3.23}$$

with a scalar coefficient. Taking $\boldsymbol{a}=\boldsymbol{a}_K$, $\boldsymbol{b}=\boldsymbol{a}_L^2$ in (2.5), we see that (3.23) can be expressed as a matrix polynomial of total degree 4 and extension 3. Combining this result with Theorem 1, we have

Corollary 2. *Any symmetric matrix polynomial in R, 3×3 symmetric matrices $\boldsymbol{a}_P$ can be expressed as a symmetric matrix polynomial of extension $\leqq R+1$ and of total degree $\leqq R+2$.*

4. The contraction of matrix polynomials in five or fewer 3×3 matrices

Matrix polynomials in two 3×3 matrices. We consider a matrix polynomial $\boldsymbol{P}$ in the two 3×3 matrices $\boldsymbol{a}$ and $\boldsymbol{b}$. It follows from Corollary 1 to Theorem 1, that $\boldsymbol{P}$ can be expressed as a matrix polynomial of total degree ≤ 5 and extension $\leqq 3$. From Theorem 1, we see that the matrix products in this polynomial may be formed from the factors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}^2$, $\boldsymbol{b}^2$ subject to certain limitations. We list below all the products of total degree $\leqq 5$ which can be formed from the factors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}^2$, $\boldsymbol{b}^2$, subject to the limitations imposed by Theorem 1, according to their total degrees:

$$\begin{array}{ll} \text{total degree } 0 & \boldsymbol{I}, \\ \text{total degree } 1 & \boldsymbol{a},\ \boldsymbol{b}, \\ \text{total degree } 2 & \boldsymbol{a}^2,\ \boldsymbol{b}^2,\ \boldsymbol{a}\boldsymbol{b},\ \boldsymbol{b}\boldsymbol{a}, \\ \text{total degree } 3 & \boldsymbol{a}^2\boldsymbol{b},\ \boldsymbol{b}^2\boldsymbol{a},\ \boldsymbol{a}\boldsymbol{b}^2,\ \boldsymbol{b}\boldsymbol{a}^2, \\ \text{total degree } 4 & \boldsymbol{a}^2\boldsymbol{b}^2,\ \boldsymbol{b}^2\boldsymbol{a}^2,\ \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2,\ \boldsymbol{b}\boldsymbol{a}\boldsymbol{b}^2, \\ \text{total degree } 5 & \boldsymbol{a}\boldsymbol{b}^2\boldsymbol{a}^2,\ \boldsymbol{b}\boldsymbol{a}^2\boldsymbol{b}^2. \end{array} \tag{4.1}$$

We now consider that $\boldsymbol{a}$ and $\boldsymbol{b}$ are symmetric 3×3 matrices and that $\boldsymbol{P}$ is a symmetric matrix polynomial in these. Then

$$\boldsymbol{P} = \tfrac{1}{2}(\boldsymbol{P} + \boldsymbol{P}'), \tag{4.2}$$

where $\boldsymbol{P}'$ is the transpose of $\boldsymbol{P}$. Now the transpose of any matrix product of symmetric matrices may be obtained by reversing the order of the factors in the product. Thus, the transpose of $\boldsymbol{a}\boldsymbol{b}$ is $\boldsymbol{b}\boldsymbol{a}$, the transpose of $\boldsymbol{a}^2\boldsymbol{b}$ is $\boldsymbol{b}\boldsymbol{a}^2$ and so on. We thus see that if $\boldsymbol{P}$ is a symmetric matrix polynomial in the two symmetric matrices $\boldsymbol{a}$ and $\boldsymbol{b}$, it may be expressed as the sum of the following terms with scalar coefficients

$$\begin{array}{c} \boldsymbol{I}, \\ \boldsymbol{a},\ \boldsymbol{b}, \\ \boldsymbol{a}^2,\ \boldsymbol{b}^2,\ \boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}, \\ \boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2,\quad \boldsymbol{a}\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}, \\ \boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2,\quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a},\quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}\boldsymbol{b}, \\ \boldsymbol{a}\boldsymbol{b}^2\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{a},\quad \boldsymbol{b}\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{b}. \end{array} \tag{4.3}$$

Now, from (2.5) and (2.4) and the relations obtained from these by replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\boldsymbol{b}$ and $\boldsymbol{a}$, by $\boldsymbol{a}$ and $\boldsymbol{b}^2$ and by $\boldsymbol{b}$ and $\boldsymbol{a}^2$, together with the Hamilton-Cayley theorem, we can express $\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a}$, $\boldsymbol{b}\boldsymbol{a}\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}\boldsymbol{b}$, $\boldsymbol{a}\boldsymbol{b}^2\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{a}$

and $\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{b}^2 + \boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{b}$ as matrix polynomials consisting of the terms $\boldsymbol{I}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{a}^2$, $\boldsymbol{b}^2$, $\boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}$, $\boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2$, $\boldsymbol{a}\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}$ and $\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2$ with scalar coefficients. We thus have the result, previously obtained in [2], that any symmetric matrix polynomial in the two symmetric, 3×3 matrices $\boldsymbol{a}$ and $\boldsymbol{b}$, may be expressed as the sum of the following terms with scalar coefficients:

$$
\begin{gathered}
\boldsymbol{I},\\
\boldsymbol{a},\quad \boldsymbol{b},\\
\boldsymbol{a}^2,\quad \boldsymbol{b}^2,\quad \boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a},\\
\boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2,\quad \boldsymbol{a}\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a},\\
\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2.
\end{gathered}
\tag{4.4}
$$

Matrix polynomials in three, 3×3 matrices. If $\boldsymbol{P}$ is a matrix polynomial in three, 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, it follows from Corollary 1, to Theorem 1 that it can be expressed as a matrix polynomial of total degree ≤ 5 and extension ≤ 4. From Theorem 1, it is seen that the matrix products in this polynomial which involve only two of the three matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ may be limited to those given in (4.1) together with those obtained from (4.1) by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$. The matrix products which involve all three matrices may be taken as those listed below, in order of their total degrees, together with those formed from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$:

$$
\begin{array}{ll}
\text{total degree 3} & \boldsymbol{a}\boldsymbol{b}\boldsymbol{c},\\
\text{total degree 4} & \boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c},\quad \boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c},\quad \boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2,\\
\text{total degree 5} & \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c},\quad \boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}^2,\quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c},\quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2,\quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2.
\end{array}
\tag{4.5}
$$

By an argument similar to that employed in discussing symmetric matrix polynomials in two symmetric matrices, we see that if $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are symmetric 3×3 matrices and $\boldsymbol{P}$ is a symmetric matrix polynomial in these, then it can be expressed as a matrix polynomial consisting of the terms

$$
\begin{gathered}
\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}\boldsymbol{a},\\
\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}^2,\quad \boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b},\quad \boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{b},\\
\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}^2,\quad \boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{c},\quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a},\\
\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{b}\boldsymbol{a},\quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{a}\boldsymbol{b},
\end{gathered}
\tag{4.6}
$$

and the terms formed from (4.4) and (4.6) by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, each of the terms having a scalar coefficient. In (4.4), the terms $\boldsymbol{b}$, $\boldsymbol{b}^2$ and $\boldsymbol{a}\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}$ can be formed from $\boldsymbol{a}$, $\boldsymbol{a}^2$ and $\boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2$ respectively, by a permutation of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. In (4.6), the terms $\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{b}$ and $\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2\boldsymbol{c}$ can be formed from $\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}^2$ and $\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}^2$ respectively by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Also, replacing $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\boldsymbol{a}$ and $\boldsymbol{c}$ respectively in equation (2.5), we can express $\boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{a}\boldsymbol{b}$ as the sum of $-(\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b})$ and a matrix polynomial of lower total degree than 5. We note also that $\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{a}+\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}$ can be formed from $\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a}$ by a permutation of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

We thus see that any symmetric matrix polynomial in the three symmetric matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ may be expressed as the sum of the following terms and terms formed from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, with scalar coefficients:

$$\begin{gathered}\boldsymbol{I}, \quad \boldsymbol{a}, \quad \boldsymbol{a}^2, \quad \boldsymbol{a}\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}, \quad \boldsymbol{a}^2\boldsymbol{b}+\boldsymbol{b}\boldsymbol{a}^2, \quad \boldsymbol{a}^2\boldsymbol{b}^2+\boldsymbol{b}^2\boldsymbol{a}^2,\\ \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}\boldsymbol{a},\\ \boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}^2, \quad \boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b},\\ \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}^2,\\ \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}+\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}.\end{gathered} \tag{4.7}$$

Matrix polynomials in four, 3×3 matrices. We now consider a matrix polynomial $\boldsymbol{P}$ in the four 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$. From the corollary to Theorem 1, we see that $\boldsymbol{P}$ can be expressed as a matrix polynomial of total degree ≤ 6 and extension ≤ 5. From Theorem 1, it is seen that the matrix products in this polynomial which involve three or fewer of the matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ may be taken as those given by (4.1) and (4.5) and the matrix products formed from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$. Also, the matrix products which involve all four matrices may be taken as those listed below, in order of their total degrees, together with those formed from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$:

$$\begin{array}{ll} \text{total degree 4} & \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d},\\ \text{total degree 5} & \boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}, \quad \boldsymbol{a}\boldsymbol{b}^2\boldsymbol{c}\boldsymbol{d}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2,\\ \text{total degree 6} & \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}\boldsymbol{d}, \quad \boldsymbol{a}\boldsymbol{b}^2\boldsymbol{c}^2\boldsymbol{d}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}^2,\\ & \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{a}^2, \quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{d}\boldsymbol{a}^2, \quad \boldsymbol{b}\boldsymbol{c}\boldsymbol{a}\boldsymbol{d}\boldsymbol{a}^2,\\ & \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{d}, \quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{d},\\ & \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{d}. \end{array} \tag{4.8}$$

By an argument similar to that employed in discussing symmetric matrix polynomials in two and three symmetric matrices, we see that if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are four symmetric 3×3 matrices and $\boldsymbol{P}$ is a symmetric matrix polynomial, then it can be expressed as a matrix polynomial consisting of the terms (4.7), the terms

$$\begin{gathered}\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}\boldsymbol{a},\\ \boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}^2, \quad \boldsymbol{a}\boldsymbol{b}^2\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}^2\boldsymbol{b}\boldsymbol{a}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2+\boldsymbol{d}^2\boldsymbol{c}\boldsymbol{b}\boldsymbol{a},\\ \boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}^2, \quad \boldsymbol{a}\boldsymbol{b}^2\boldsymbol{c}^2\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}^2\boldsymbol{b}^2\boldsymbol{a}, \quad \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}^2+\boldsymbol{d}^2\boldsymbol{c}^2\boldsymbol{b}\boldsymbol{a},\\ \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}, \quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{d}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{d}\boldsymbol{c}\boldsymbol{a}\boldsymbol{b}, \quad \boldsymbol{b}\boldsymbol{c}\boldsymbol{a}\boldsymbol{d}\boldsymbol{a}^2+\boldsymbol{a}^2\boldsymbol{d}\boldsymbol{a}\boldsymbol{c}\boldsymbol{b},\\ \boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{d}+\boldsymbol{d}\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}, \quad \boldsymbol{b}\boldsymbol{a}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{d}+\boldsymbol{d}\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{a}\boldsymbol{b},\\ \boldsymbol{a}\boldsymbol{b}\boldsymbol{a}^2\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{a},\end{gathered} \tag{4.9}$$

and the terms formed from (4.7) and (4.9) by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, each of the terms having a scalar coefficient.

In (4.9), the terms $\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}^2+\boldsymbol{d}^2\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}$, $\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}^2\boldsymbol{b}\boldsymbol{a}$ and $\boldsymbol{a}\boldsymbol{b}\boldsymbol{c}^2\boldsymbol{d}^2+\boldsymbol{d}^2\boldsymbol{c}^2\boldsymbol{b}\boldsymbol{a}$ can be obtained from the terms $\boldsymbol{a}^2\boldsymbol{b}\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}\boldsymbol{a}^2$, $\boldsymbol{a}\boldsymbol{b}^2\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}$ and $\boldsymbol{a}^2\boldsymbol{b}^2\boldsymbol{c}\boldsymbol{d}+\boldsymbol{d}\boldsymbol{c}\boldsymbol{b}^2\boldsymbol{a}^2$ respectively by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$. Also, by employing relations obtained from (2.5) by appropriate substitutions for $\boldsymbol{a}$ and $\boldsymbol{b}$, we can obtain

equations of the form

$$\boldsymbol{a\,b\,c\,a^2 d + d\,a^2 c\,b\,a} = -(\boldsymbol{a^2 b\,c\,a\,d + d\,a\,c\,b\,a^2}) + \boldsymbol{Q}_1,$$
$$\boldsymbol{a\,b\,a^2 c\,d + d\,c\,a^2 b\,a} = -(\boldsymbol{a^2 b\,a\,c\,d + d\,c\,a\,b\,a^2}) + \boldsymbol{Q}_2,$$

where $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ are matrix polynomials of total degree <6. We note that $\boldsymbol{a^2bcad + dacba^2}$ and $\boldsymbol{a^2bacd + dcaba^2}$ can be obtained from $\boldsymbol{bacda^2 + a^2dcab}$ and $\boldsymbol{bcada^2 + a^2dacb}$ respectively by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$.

We thus see that any symmetric matrix polynomial in the four symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ may be expressed as a matrix polynomial consisting of the terms (4.7), the terms

$$\begin{gathered}\boldsymbol{a\,b\,c\,d + d\,c\,b\,a},\\ \boldsymbol{a^2 b\,c\,d + d\,c\,b\,a^2},\quad \boldsymbol{a\,b^2 c\,d + d\,c\,b^2 a},\\ \boldsymbol{a^2 b^2 c\,d + d\,c\,b^2 a^2},\quad \boldsymbol{a\,b^2 c^2 d + d\,c^2 b^2 a},\\ \boldsymbol{a\,b\,c\,d\,a^2 + a^2 d\,c\,b\,a},\quad \boldsymbol{b\,a\,c\,d\,a^2 + a^2 d\,c\,a\,b},\quad \boldsymbol{b\,c\,a\,d\,a^2 + a^2 d\,a\,c\,b},\\ \boldsymbol{b\,a\,c\,a^2 d + d\,a^2 c\,a\,b},\end{gathered} \tag{4.10}$$

and the terms formed from (4.7) and (4.10) by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$, each of these terms having a scalar coefficient.

Matrix polynomials in five, 3×3 matrices. Finally, we assume that $\boldsymbol{P}$ is a matrix polynomial in the five, 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$. As in the previous case, we see that $\boldsymbol{P}$ can be expressed as a matrix polynomial of total degree $\leqq 7$ and extension $\leqq 6$, in which the matrix products are those given by (4.1), (4.5) and (4.8) and by (4.11) below, together with the matrix products formed from these by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$.

$$\begin{array}{ll} \text{total degree 5} & \boldsymbol{a\,b\,c\,d\,e},\\ \text{total degree 6} & \boldsymbol{a^2 b\,c\,d\,e},\quad \boldsymbol{a\,b^2 c\,d\,e},\quad \boldsymbol{a\,b\,c^2 d\,e},\quad \boldsymbol{a\,b\,c\,d^2 e},\quad \boldsymbol{a\,b\,c\,d\,e^2},\\ \text{total degree 7} & \boldsymbol{a^2 b^2 c\,d\,e},\quad \boldsymbol{a\,b^2 c^2 d\,e},\quad \boldsymbol{a\,b\,c^2 d^2 e},\quad \boldsymbol{a\,b\,c\,d^2 e^2},\\ & \boldsymbol{a\,b\,c\,d\,e\,a^2},\quad \boldsymbol{a\,b\,c\,d\,a^2 e},\quad \boldsymbol{a\,b\,c\,a^2 d\,e},\quad \boldsymbol{a\,b\,a^2 c\,d\,e},\\ & \boldsymbol{b\,a\,c\,d\,e\,a^2},\quad \boldsymbol{b\,a\,c\,d\,a^2 e},\quad \boldsymbol{b\,a\,c\,a^2 d\,e},\\ & \boldsymbol{b\,c\,a\,d\,e\,a^2},\quad \boldsymbol{b\,c\,a\,d\,a^2 e},\\ & \boldsymbol{b\,c\,d\,a\,e\,a^2}.\end{array} \tag{4.11}$$

We now assume that $\boldsymbol{P}$ is a symmetric matrix polynomial in five symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$. Then, in a manner similar to that employed in discussing symmetric matrix polynomials in four 3×3 matrices, we can show that $\boldsymbol{P}$ may be expressed as a matrix polynomial, the terms in which are those listed in (4.7) and (4.10), the terms

$$\begin{gathered}\boldsymbol{a\,b\,c\,d\,e + e\,d\,c\,b\,a},\\ \boldsymbol{a^2 b\,c\,d\,e + e\,d\,c\,b\,a^2},\quad \boldsymbol{a\,b^2 c\,d\,e + e\,d\,c\,b^2 a},\quad \boldsymbol{a\,b\,c^2 d\,e + e\,d\,c^2 b\,a},\\ \boldsymbol{a^2 b^2 c\,d\,e + e\,d\,c\,b^2 a^2},\quad \boldsymbol{a\,b^2 c^2 d\,e + e\,d\,c^2 b^2 a},\\ \boldsymbol{a\,b\,c\,d\,e\,a^2 + a^2 e\,d\,c\,b\,a},\ \boldsymbol{a\,b\,c\,d\,a^2 e + e\,a^2 d\,c\,b\,a},\ \boldsymbol{a\,b\,c\,a^2 d\,e + e\,d\,a^2 c\,b\,a},\\ \boldsymbol{a\,b\,a^2 c\,d\,e + e\,d\,c\,a^2 b\,a},\ \boldsymbol{b\,a\,c\,d\,a^2 e + e\,a^2 d\,c\,a\,b},\ \boldsymbol{b\,a\,c\,a^2 d\,e + e\,d\,a^2 c\,a\,b},\end{gathered} \tag{4.12}$$

and the terms formed from (4.7), (4.10) and (4.12) by permutations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$, each of these terms having a scalar coefficient.

5. Scalar invariants under the orthogonal group of R symmetric 3×3 matrices

In this and the following four sections, we shall discuss the determination of integrity bases for absolute invariants of R symmetric 3×3 matrices under the group of orthogonal transformations. It will be seen that both the argument and results are independent of whether it is the full or proper orthogonal group that is considered. Since we shall be concerned only with absolute invariants, we shall, for brevity, generally omit the qualification "absolute".

Let us define the elements of R, 3×3 (not necessarily symmetric) matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ by

$$\boldsymbol{a}_P = \left\| a_{ij}^{(P)} \right\|. \tag{5.1}$$

Any matrix product $\boldsymbol{\Pi}$ of the R matrices $\boldsymbol{a}_P$ may be expressed in the form

$$\boldsymbol{\Pi} = \boldsymbol{a}_M \boldsymbol{\Pi}^*, \tag{5.2}$$

where $\boldsymbol{\Pi}^*$ is itself a matrix product of the matrices $\boldsymbol{a}_P$ of total degree one less than that of $\boldsymbol{\Pi}$.

Now, from Theorem 1, the matrix product $\boldsymbol{\Pi}^*$ may be expressed as a matrix polynomial of lower or equal partial degrees in each of the matrices and of lower or equal extension, in which each matrix product takes one of the forms

$$\begin{aligned} &\boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \quad (K \neq L), \\ &\boldsymbol{y}\, \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} \quad (K \neq L), \\ &\boldsymbol{v}\, \boldsymbol{a}_K^2 \boldsymbol{w}, \end{aligned} \tag{5.3}$$

or

$$\boldsymbol{u},$$

where $\boldsymbol{y}$ and $\boldsymbol{z}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R;\ P\neq K, L)$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{u}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$, no two factors in $\boldsymbol{y}\boldsymbol{z}$, in $\boldsymbol{v}\boldsymbol{w}$ or in $\boldsymbol{u}$ being the same. $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{u}$ may, as particular cases, be the unit matrix $\boldsymbol{I}$. Thus, denoting the matrix products (5.3) by $\widetilde{\boldsymbol{\omega}}_\alpha$ $(\alpha=1, 2, \ldots)$, we have

$$\boldsymbol{\Pi}^* = \sum_\alpha \chi_\alpha \widetilde{\boldsymbol{\omega}}_\alpha, \tag{5.4}$$

where the coefficients χ_α are scalars, which, from the manner in which Theorem 1 is derived, are readily seen to be expressible as polynomials in traces of matrix products formed from $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ of partial degrees in each of the matrices less than or equal to those of $\boldsymbol{\Pi}^*$.

From (5.2) and (5.4), we obtain

$$\boldsymbol{\Pi} = \sum_\alpha \chi_\alpha \boldsymbol{a}_M \widetilde{\boldsymbol{\omega}}_\alpha. \tag{5.5}$$

Thus,

$$\operatorname{tr} \boldsymbol{\Pi} = \sum_\alpha \chi_\alpha \operatorname{tr} \boldsymbol{a}_M \widetilde{\boldsymbol{\omega}}_\alpha. \tag{5.6}$$

By successively repeating this procedure for each of the traces of matrix products in terms of which the coefficients χ_α are expressed, it is seen that $\operatorname{tr}\boldsymbol{\Pi}$ must be expressible as a polynomial in the expressions $\operatorname{tr} \boldsymbol{a}_M \widetilde{\boldsymbol{\omega}}_\alpha$. We thus obtain

Lemma 4. *The trace of a matrix product formed from the R, 3×3 matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ may be expressed as a polynomial in expressions having the forms*

$$\operatorname{tr} \boldsymbol{a}_M \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \quad (K \neq L), \tag{5.7}$$

$$\operatorname{tr} \boldsymbol{a}_M \boldsymbol{y} \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} \quad (K \neq L), \tag{5.8}$$

$$\operatorname{tr} \boldsymbol{a}_M \boldsymbol{v} \boldsymbol{a}_K^2 \boldsymbol{w} \tag{5.9}$$

and

$$\operatorname{tr} \boldsymbol{a}_m \boldsymbol{u}, \tag{5.10}$$

which have partial degrees in each of the matrices $\boldsymbol{a}_P$ less than or equal to the partial degrees of $\operatorname{tr} \boldsymbol{\Pi}$, *where $\boldsymbol{y}$ and $\boldsymbol{z}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R;\ P \neq K, L)$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{u}$ are matrix products formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$, no two factors in $\boldsymbol{y}\boldsymbol{z}$, in $\boldsymbol{v}\boldsymbol{w}$ or in $\boldsymbol{u}$ being the same, and, in particular cases $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and $\boldsymbol{u}$ may be the unit matrix $\boldsymbol{I}$.*

We note that the trace of any matrix product of R matrices $\boldsymbol{a}_P$ is a scalar invariant of the matrices under all orthogonal transformations. In particular, the expressions (5.7), (5.8), (5.9) and (5.10) are scalar invariants.

In the case when the R matrices $\boldsymbol{a}_P$ are symmetric, we have [2]

Lemma 5. *Any polynomial scalar invariant, under either the full or the proper orthogonal group, of any number of symmetric matrices may be expressed as a polynomial in traces of matrix products formed from these matrices.*

It follows, of course, from Lemmas 4 and 5 that the set of invariants (5.7), (5.8), (5.9) and (5.10) forms a finite integrity basis for the R symmetric 3×3 matrices $\boldsymbol{a}_P$ under either the full or the proper orthogonal group.

6. Some relations concerning the reducibility of scalar invariants of symmetric 3×3 matrices

If I_1 and I_2 are two scalar invariants for the set of R (not necessarily symmetric) 3×3 matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ of equal partial degrees in each of the matrices and I_1 may be expressed in the form

$$I_1 = I_2 + P, \tag{6.1}$$

where P is a polynomial in invariants of total degree less than that of I_1 and I_2, we say that I_1 is equivalent to I_2 and write

$$I_1 \equiv I_2. \tag{6.2}$$

$I_1 - I_2$ is itself a scalar invariant of the set of matrices and is said to be *reducible*. We indicate this by

$$I_1 - I_2 \equiv 0. \tag{6.3}$$

In the present paper, we shall derive a number of relations between traces of matrix products formed from the R matrices $\boldsymbol{a}_P$, which are of value in establishing the equivalence between various invariants of the matrices and the reducibility of others.

It is easily seen that

$$\operatorname{tr} \boldsymbol{a}_{i_1}^{\alpha_1} \boldsymbol{a}_{i_2}^{\alpha_2} \ldots \boldsymbol{a}_{i_e}^{\alpha_e} = \operatorname{tr} \boldsymbol{a}_{i_e}^{\alpha_e} \boldsymbol{a}_{i_1}^{\alpha_1} \boldsymbol{a}_{i_2}^{\alpha_2} \ldots \boldsymbol{a}_{i_{e-1}}^{\alpha_{e-1}} = \operatorname{tr} \boldsymbol{a}_{i_{e-1}}^{\alpha_{e-1}} \boldsymbol{a}_{i_e}^{\alpha_e} \boldsymbol{a}_{i_1}^{\alpha_1} \ldots \boldsymbol{a}_{i_{e-2}}^{\alpha_{e-2}} = \cdots, \tag{6.4}$$

where each of the subscripts $i_1, i_2, \ldots, i_e$ is one of the numbers $1, 2, \ldots, R$ and the indices $\alpha_1, \alpha_2, \ldots, \alpha_e$ are positive integers. We thus have

Lemma 6. *The trace of a matrix product formed from 3×3 matrices is unaltered by cyclic permutation of the factors in the product.*

Now, suppose that the matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ are symmetric 3×3 matrices. We see that $\boldsymbol{a}_{i_e}^{\alpha_e} \ldots \boldsymbol{a}_{i_2}^{\alpha_2} \boldsymbol{a}_{i_1}^{\alpha_1}$ is the transpose of $\boldsymbol{a}_{i_1}^{\alpha_1} \boldsymbol{a}_{i_2}^{\alpha_2} \ldots \boldsymbol{a}_{i_e}^{\alpha_e}$. Since the trace of any matrix is equal to the trace of its transpose, we have

Lemma 7. *The trace of a matrix product formed from symmetric 3×3 matrices is unaltered if the order of the factors in the product is reversed.*

A number of further relations between traces of matrix products formed from 3×3 matrices can be obtained from the relations derived in § 2. Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three 3×3 matrices and let $\boldsymbol{y}$ and $\boldsymbol{z}$ be any 3×3 matrices (in particular cases $\boldsymbol{I}$). Then multiplying (2.2) throughout on the left by $\boldsymbol{y}$ and on the right by $\boldsymbol{z}$, we obtain

$$\begin{aligned} &\operatorname{tr} \boldsymbol{y\,a\,b\,c\,z} + \operatorname{tr} \boldsymbol{y\,b\,c\,a\,z} + \operatorname{tr} \boldsymbol{y\,c\,a\,b\,z} + \\ &\qquad + \operatorname{tr} \boldsymbol{y\,b\,a\,c\,z} + \operatorname{tr} \boldsymbol{y\,a\,c\,b\,z} + \operatorname{tr} \boldsymbol{y\,c\,b\,a\,z} \equiv 0, \end{aligned} \tag{6.5}$$

unless $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

Taking $\boldsymbol{a}=\boldsymbol{b}=\boldsymbol{c}$ in (6.5), we obtain

$$\operatorname{tr} \boldsymbol{y\,a}^3 \boldsymbol{z} \equiv 0, \tag{6.6}$$

unless $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

Again, taking $\boldsymbol{c}=\boldsymbol{a}$ in (6.5), we obtain

$$\operatorname{tr} \boldsymbol{y\,a\,b\,a\,z} + \operatorname{tr} \boldsymbol{y\,b\,a}^2 \boldsymbol{z} + \operatorname{tr} \boldsymbol{y\,a}^2 \boldsymbol{b\,z} \equiv 0, \tag{6.7}$$

unless $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

From equation (2.5), we obtain

$$\operatorname{tr} \boldsymbol{y\,a\,b\,a}^2 \boldsymbol{z} + \operatorname{tr} \boldsymbol{y\,a}^2 \boldsymbol{b\,a\,z} \equiv 0, \tag{6.8}$$

even if $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

From equation (2.6), we obtain

$$\operatorname{tr} \boldsymbol{y\,a}^2 \boldsymbol{b\,a}^2 \boldsymbol{z} \equiv 0, \tag{6.9}$$

even if $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

Now, let $\boldsymbol{x}$ $(\neq \boldsymbol{I})$ be any 3×3 matrix. Multiplying (2.15) throughout on the left by $\boldsymbol{y}$ and on the right by $\boldsymbol{z}$, we have

$$\operatorname{tr} \boldsymbol{y\,a}^2 \boldsymbol{x\,b}^2 \boldsymbol{z} \equiv 0, \tag{6.10}$$

unless $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

Multiplying (2.15) throughout on the right by $\boldsymbol{y\,c}^2$, we obtain

$$\operatorname{tr} \boldsymbol{a}^2 \boldsymbol{x\,b}^2 \boldsymbol{y\,c}^2 \equiv 0. \tag{6.11}$$

Again multiplying (2.20) throughout on the left by $\boldsymbol{y}$ and on the right by $\boldsymbol{z}$, we obtain

$$\operatorname{tr} \boldsymbol{y\,a}^2 \boldsymbol{b}^2 \boldsymbol{c}^2 \boldsymbol{z} \equiv 0, \tag{6.12}$$

unless $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$.

If $\boldsymbol{y}=\boldsymbol{z}=\boldsymbol{I}$, on taking the trace of both sides of (2.20), we have, with Lemma 6,

$$\operatorname{tr} \boldsymbol{a}^2 \boldsymbol{b}^2 \boldsymbol{c}^2 + \operatorname{tr} \boldsymbol{c}^2 \boldsymbol{b}^2 \boldsymbol{a}^2 \equiv 0. \tag{6.13}$$

If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are symmetric matrices, we obtain from (6.13) and Lemma 7

$$\operatorname{tr} \boldsymbol{a}^2 \boldsymbol{b}^2 \boldsymbol{c}^2 \equiv 0. \tag{6.14}$$

Finally, multiplying equations (2.28) and (2.29) on the left by $\boldsymbol{z}$, we obtain

$$\operatorname{tr} \boldsymbol{z}\,\boldsymbol{a}\,\boldsymbol{b}^2 \boldsymbol{a}^2 \boldsymbol{y} \equiv 0 \tag{6.15}$$

and

$$\operatorname{tr} \boldsymbol{z}\,\boldsymbol{a}^2 \boldsymbol{b}^2 \boldsymbol{a}\,\boldsymbol{y} \equiv 0, \tag{6.16}$$

unless $\boldsymbol{y}=\boldsymbol{I}$ or $\boldsymbol{z}=\boldsymbol{I}$.

7. The reduction of the invariants for R, 3×3 matrices

In this section, we shall use the results of § 6 to investigate the reducibility of the invariants (5.7), (5.8), (5.9) and (5.10).

We first consider invariants of the form (5.7), *viz.* $\operatorname{tr} \boldsymbol{a}_M \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2$. As a special case, we may have $M=L$, but if $M=K$, the invariant is $\operatorname{tr} \boldsymbol{a}_K^2 \boldsymbol{a}_L^2 \boldsymbol{a}_K^2$, which by Lemma 6 is $\operatorname{tr} \boldsymbol{a}_K^4 \boldsymbol{a}_L^2$ and is therefore reducible.

Next we consider invariants of the form (5.8). These have total degree $\leqq R+3$. We consider the three cases $M=K$, $M=L$, $M \neq K, L$ separately.

Case $M=K$. We have, from (6.8),

$$\operatorname{tr} \boldsymbol{a}_K \boldsymbol{y}\,\boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} \equiv -\operatorname{tr} \boldsymbol{a}_K^2 \boldsymbol{y}\,\boldsymbol{a}_L^2 \boldsymbol{a}_K \boldsymbol{z},$$

which, from (6.10), is reducible unless $\boldsymbol{y}=\boldsymbol{I}$. If $\boldsymbol{y}=\boldsymbol{I}$, (5.8) takes the form $\operatorname{tr} \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z}$. Then, either $\boldsymbol{z}=\boldsymbol{I}$ so that, using Lemma 6, (5.8) may be written $\operatorname{tr} \boldsymbol{a}_K^3 \boldsymbol{a}_L^2$, which is reducible; or $\boldsymbol{z}=\boldsymbol{a}_P$ and (5.8) becomes $\operatorname{tr} \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{a}_P$, which may be written as $\operatorname{tr} \boldsymbol{a}_P \boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2$ and then has the form (5.7) discussed above; or, finally, $\boldsymbol{z}=\boldsymbol{a}_P \boldsymbol{w}$ $(\boldsymbol{w} \neq \boldsymbol{I})$. In this latter case, using Lemma 6, (5.8) may be expressed in the form $\operatorname{tr} \boldsymbol{w}\,\boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{a}_P$. Replacing $\boldsymbol{z}$, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{y}$ by $\boldsymbol{w}$, $\boldsymbol{a}_K$, $\boldsymbol{a}_L$ and $\boldsymbol{a}_P$ respectively in (6.15), we see that $\operatorname{tr} \boldsymbol{w}\,\boldsymbol{a}_K \boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{a}_P$ $(\boldsymbol{w} \neq \boldsymbol{I})$ is reducible.

Case $M=L$. We have, from (6.8),

$$\operatorname{tr} \boldsymbol{a}_L \boldsymbol{y}\,\boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} \equiv -\operatorname{tr} \boldsymbol{a}_L^2 \boldsymbol{y}\,\boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{z},$$

which is reducible unless $\boldsymbol{z}=\boldsymbol{I}$. Since, from Lemma 6, $\operatorname{tr} \boldsymbol{a}_L^2 \boldsymbol{y}\,\boldsymbol{a}_L \boldsymbol{a}_K^2 = \operatorname{tr} \boldsymbol{y}\,\boldsymbol{a}_L \boldsymbol{a}_K^2 \boldsymbol{a}_L^2$, we see, as before, that $\operatorname{tr} \boldsymbol{a}_L \boldsymbol{y}\,\boldsymbol{a}_L^2 \boldsymbol{a}_K^2 \boldsymbol{z} \equiv 0$ unless $\boldsymbol{z}=\boldsymbol{I}$ and $\boldsymbol{y}=\boldsymbol{a}_P$.

Case $M \neq K, L$. If $\boldsymbol{a}_M$ is not a factor of $\boldsymbol{y}$ or $\boldsymbol{z}$, then (5.8) must of necessity have total degree $\leqq R+2$ and must be expressible in the form $\operatorname{tr} \boldsymbol{y}\boldsymbol{a}_L^2 \boldsymbol{a}_K^2$ where $\boldsymbol{y}$ is a matrix product formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R;$ $P \neq K, L)$, no factor being repeated. If $\boldsymbol{a}_M$ is a factor of either $\boldsymbol{y}$ or $\boldsymbol{z}$, it is easily seen that (5.8) is reducible, by using one or more of the results expressed by (6.7), (6.10), (6.12) and Lemma 6, depending on the position of the factor $\boldsymbol{a}_M$ in $\boldsymbol{y}$ or $\boldsymbol{z}$.

We now consider invariants of the form (5.9). These have total degree $\leqq R+3$. If $\boldsymbol{a}_M$ is not a factor of either $\boldsymbol{v}$ or $\boldsymbol{w}$, (5.9) may be expressed in the form $\operatorname{tr} \boldsymbol{v}\boldsymbol{a}_K^2$, where $\boldsymbol{v}$ is a matrix product formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$, no factor being repeated. If $\boldsymbol{a}_M$ is a factor of $\boldsymbol{v}$ or $\boldsymbol{w}$, then either $M=K$, or $M \neq K$. We shall treat these two cases separately.

Case $M=K$. If $\boldsymbol{a}_K$ is a factor of $\boldsymbol{v}$, the invariant (5.9) takes either the form $\operatorname{tr}\boldsymbol{a}_K^2\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}$ $(\boldsymbol{u}\neq\boldsymbol{I})$ or $\operatorname{tr}\boldsymbol{a}_K\boldsymbol{x}\boldsymbol{a}_K\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}$ $(\boldsymbol{x}\neq\boldsymbol{I},\boldsymbol{u}\neq\boldsymbol{I})$. From (6.9), we see that the first of these forms is reducible. We have, from (6.7).

$$\operatorname{tr}\boldsymbol{a}_K\boldsymbol{x}\boldsymbol{a}_K\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}\equiv-\operatorname{tr}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}-\operatorname{tr}\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w},$$

and, from (6.9), each of the expressions on the right-hand side is reducible. If $\boldsymbol{a}_K$ is a factor of $\boldsymbol{w}$, the invariant (5.9) takes the form $\operatorname{tr}\boldsymbol{a}_K\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_K\boldsymbol{u}$ $(\boldsymbol{x}\neq\boldsymbol{I})$. This is seen to be reducible in a manner similar to that employed in discussing the reducibility of $\operatorname{tr}\boldsymbol{a}_K\boldsymbol{x}\boldsymbol{a}_K\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}$.

Case $M\neq K$. If $\boldsymbol{a}_M$ is a factor of $\boldsymbol{v}$, then (5.9) takes either the form $\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{w}$ or $\operatorname{tr}\boldsymbol{a}_M\boldsymbol{u}\boldsymbol{a}_M\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{w}$ $(\boldsymbol{u}\neq\boldsymbol{I})$. In the first case the form is reducible, from (6.10), unless $\boldsymbol{x}=\boldsymbol{I}$ or $\boldsymbol{w}=\boldsymbol{I}$, in which case it may be written $\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{a}_K^2\boldsymbol{w}$ or $\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{x}\boldsymbol{a}_K^2$, which from Lemma 6 is $\operatorname{tr}\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{a}_M^2$. In the second case, we have, from (6.7),

$$\operatorname{tr}\boldsymbol{a}_M\boldsymbol{u}\boldsymbol{a}_M\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{w}\equiv-\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{u}\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{w}-\operatorname{tr}\boldsymbol{u}\boldsymbol{a}_M^2\boldsymbol{x}\boldsymbol{a}_K^2\boldsymbol{w}$$

and from (6.10) we see that both terms on the right-hand side are reducible provided that $\boldsymbol{x}\neq\boldsymbol{I}$ and $\boldsymbol{w}\neq\boldsymbol{I}$. If $\boldsymbol{x}=\boldsymbol{I}$, the first term on the right-hand side takes the form $-\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}$ $(\boldsymbol{u}\neq\boldsymbol{I})$ and the second term takes the form $-\operatorname{tr}\boldsymbol{u}\boldsymbol{a}_M^2\boldsymbol{a}_K^2\boldsymbol{w}$ $(\boldsymbol{u}\neq\boldsymbol{I})$. Now, $\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{w}$ is reducible unless $\boldsymbol{w}=\boldsymbol{I}$ in which case it takes the form $\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{u}\boldsymbol{a}_K^2$, which from Lemma 6 is $\operatorname{tr}\boldsymbol{u}\boldsymbol{a}_K^2\boldsymbol{a}_M^2$. Also, from Lemma 6, $\operatorname{tr}\boldsymbol{u}\boldsymbol{a}_M^2\boldsymbol{a}_K^2\boldsymbol{w}=\operatorname{tr}\boldsymbol{w}\boldsymbol{u}\boldsymbol{a}_M^2\boldsymbol{a}_K^2$. If $\boldsymbol{a}_M$ is a factor of $\boldsymbol{w}$, then (5.9) takes the form $\operatorname{tr}\boldsymbol{a}_M\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_M\boldsymbol{u}$, and, from (6.7), we have

$$\operatorname{tr}\boldsymbol{a}_M\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_M\boldsymbol{u}\equiv-\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{u}-\operatorname{tr}\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_M^2\boldsymbol{u}.$$

From (6.10) the first of these terms is reducible unless $\boldsymbol{v}=\boldsymbol{I}$ or $\boldsymbol{x}\boldsymbol{u}=\boldsymbol{I}$, in which case it takes either the form $-\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{u}$ or $-\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{v}\boldsymbol{a}_K^2$, which from Lemma 6 are $-\operatorname{tr}\boldsymbol{x}\boldsymbol{u}\boldsymbol{a}_M^2\boldsymbol{a}_K^2$ and $-\operatorname{tr}\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{a}_M^2$. From (6.10) $\operatorname{tr}\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_M^2\boldsymbol{u}$ is reducible unless $\boldsymbol{x}=\boldsymbol{I}$ or both $\boldsymbol{v}=\boldsymbol{I}$ and $\boldsymbol{u}=\boldsymbol{I}$, in which cases it takes either the form $\operatorname{tr}\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{a}_M^2\boldsymbol{u}$ or $\operatorname{tr}\boldsymbol{a}_K^2\boldsymbol{x}\boldsymbol{a}_M^2$, which, from Lemma 6, are equal to $\operatorname{tr}\boldsymbol{u}\boldsymbol{v}\boldsymbol{a}_K^2\boldsymbol{a}_M^2$ and $\operatorname{tr}\boldsymbol{x}\boldsymbol{a}_M^2\boldsymbol{a}_K^2$.

We thus see that invariants of the form (5.9) are either reducible or equivalent to linear combinations of invariants of the forms $\operatorname{tr}\boldsymbol{v}\boldsymbol{a}_K^2$ and $\operatorname{tr}\boldsymbol{y}\boldsymbol{a}_L^2\boldsymbol{a}_K^2$, where $\boldsymbol{v}$ is a matrix product formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1,2,\ldots,R)$ and $\boldsymbol{y}$ is a matrix product formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1,2,\ldots,R;\ P\neq K,L)$, no factors in $\boldsymbol{v}$ or in $\boldsymbol{y}$ being repeated.

Finally, we consider invariants of the form (5.10). These have total degree $\leqq R+1$. If $\boldsymbol{a}_M$ is not a factor of $\boldsymbol{u}$, then (5.10) has the form $\operatorname{tr}\boldsymbol{u}$, where $\boldsymbol{u}$ is a matrix product formed from some or all of the factors $\boldsymbol{a}_P$ $(P=1,2,\ldots,R)$, no factor being repeated. If $\boldsymbol{a}_M$ is a factor $\boldsymbol{u}$ then (5.10) has the form $\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{v}$ or $\operatorname{tr}\boldsymbol{a}_M\boldsymbol{x}\boldsymbol{a}_M\boldsymbol{y}$ $(\boldsymbol{x}\neq\boldsymbol{I})$. In the latter case we see, from (6.7) and Lemma 6, that

$$\operatorname{tr}\boldsymbol{a}_M\boldsymbol{x}\boldsymbol{a}_M\boldsymbol{y}\equiv-\operatorname{tr}\boldsymbol{a}_M^2\boldsymbol{x}\boldsymbol{y}-\operatorname{tr}\boldsymbol{x}\boldsymbol{a}_M^2\boldsymbol{y}=-\operatorname{tr}\boldsymbol{x}\boldsymbol{y}\boldsymbol{a}_M^2-\operatorname{tr}\boldsymbol{y}\boldsymbol{x}\boldsymbol{a}_M^2.$$

Combining the results obtained in this section with Lemma 4, we obtain

Theorem 2. *The trace of any matrix product formed from the* R, 3×3 *matrices* $\boldsymbol{a}_P$ $(P=1,2,\ldots,R)$ *may be expressed as a polynomial in traces of matrix products of lower or equal partial degrees in each of the matrices and of lower or equal*

extensions, having the forms

$$\operatorname{tr} \boldsymbol{y}\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L^2 \quad (K \neq L), \tag{7.1}$$

$$\operatorname{tr} \boldsymbol{v}\, \boldsymbol{a}_K^2, \tag{7.2}$$

$$\operatorname{tr} \boldsymbol{u}, \tag{7.3}$$

$$\operatorname{tr} \boldsymbol{a}_M\, \boldsymbol{a}_L\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L^2 \quad (K \neq L,\ M \neq L) \tag{7.4}$$

and

$$\operatorname{tr} \boldsymbol{a}_K^3, \tag{7.5}$$

where $\boldsymbol{y}$ *is either the unit matrix or a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$; $P \neq K, L$), *no two factors in* $\boldsymbol{y}$ *being the same;* $\boldsymbol{v}$ *is either the unit matrix or a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), *such that* $\boldsymbol{a}_K$ *is not the first or last factor in* $\boldsymbol{v}$ *and no two factors in* $\boldsymbol{v}$ *are the same;* $\boldsymbol{u}$ *is a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), *no two factors in which are the same.*

We shall now assume that each of the matrices $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$) is symmetric. Then, from Lemmas 6 and 7, we have

$$\operatorname{tr} \boldsymbol{a}_M\, \boldsymbol{a}_L\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L^2 = \operatorname{tr} \boldsymbol{a}_L^2\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L\, \boldsymbol{a}_M = \operatorname{tr} \boldsymbol{a}_M\, \boldsymbol{a}_L^2\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L .$$

Using the relation (6.8), we have

$$\operatorname{tr} \boldsymbol{a}_M\, \boldsymbol{a}_L\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L^2 = \tfrac{1}{2} \operatorname{tr} \boldsymbol{a}_M (\boldsymbol{a}_L\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L^2 + \boldsymbol{a}_L^2\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L) \equiv 0 .$$

We thus obtain

Theorem 3. *An integrity basis for the* R *symmetric* 3×3 *matrices* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), *under the full or the proper orthogonal group, is formed by the invariants, of total degree* $\leq R+2$,

$$\operatorname{tr} \boldsymbol{z}\, \boldsymbol{a}_K^2\, \boldsymbol{a}_L^2 \quad (K \neq L), \tag{7.6}$$

$$\operatorname{tr} \boldsymbol{w}\, \boldsymbol{a}_K^2, \tag{7.7}$$

$$\operatorname{tr} \boldsymbol{x} \tag{7.8}$$

and

$$\operatorname{tr} \boldsymbol{a}_K^3, \tag{7.9}$$

where $\boldsymbol{z}$ *is either the unit matrix or a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$; $P \neq K, L$), *no two factors in* $\boldsymbol{z}$ *being the same;* $\boldsymbol{w}$ *is either the unit matrix or a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), *no two factors in* $\boldsymbol{w}$ *being the same and if* $\boldsymbol{a}_K$ *is one of the factors of* $\boldsymbol{w}$, *it is not the first factor or the last factor;* $\boldsymbol{x}$ *is a matrix product formed from some or all of the factors* $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), *no two factors in* $\boldsymbol{x}$ *being the same.*

8. Integrity bases for five or fewer symmetric matrices

From Theorem 3 we may write down directly finite integrity bases for five or fewer symmetric matrices under the orthogonal group.

It is evident that if we write down all the invariants given by equations (7.6), (7.7), (7.8) and (7.9) for R symmetric matrices $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), these consist of the invariants for each set of $R-1$ matrices which can be chosen from $\boldsymbol{a}_P$ ($P=1, 2, \ldots, R$), together with the invariants given by (7.6), (7.7) and (7.8) which involve all R of the matrices. Thus an integrity basis for five symmetric 3×3 matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}, \boldsymbol{e}$ may be formed by the integrity bases for each set of

four matrices which can be selected from $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, together with the invariants given by equations (7.6), (7.7) and (7.8) which involve all five of the matrices.

Single matrix $\boldsymbol{a}$. From (7.8), (7.7) and (7.9) respectively, we obtain only the invariants

$$\operatorname{tr}\boldsymbol{a}, \quad \operatorname{tr}\boldsymbol{a}^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{a}^3, \tag{8.1}$$

which therefore form an integrity basis for the single matrix $\boldsymbol{a}$. We note that no invariants of a single matrix are given by (7.6).

Two matrices $\boldsymbol{a}$ *and* $\boldsymbol{b}$. Writing down only the invariants given by (7.6), (7.7) and (7.8) by taking $\boldsymbol{a}_K = \boldsymbol{a}$ and $\boldsymbol{a}_L = \boldsymbol{b}$, which involve both $\boldsymbol{a}$ and $\boldsymbol{b}$, we obtain

$$\operatorname{tr}\boldsymbol{a}^2\boldsymbol{b}^2, \quad \operatorname{tr}\boldsymbol{b}\,\boldsymbol{a}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}, \quad \operatorname{tr}\boldsymbol{b}\,\boldsymbol{a}. \tag{8.2}$$

Thus, an integrity basis for the two symmetric 3×3 matrices $\boldsymbol{a}$ and $\boldsymbol{b}$ under the orthogonal group is formed by the invariants (8.1) and (8.2), together with the invariants formed from these by interchange of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Three matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. The invariants given by Theorem 3 which involve all three matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are

$$\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{c}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{c}^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}, \tag{8.3}$$

together with the invariants obtained from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Now, employing Lemmas 6 and 7, we have

$$\operatorname{tr}\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{c}^2 = \operatorname{tr}\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{c}^2\boldsymbol{a} = \operatorname{tr}\boldsymbol{a}\,\boldsymbol{c}^2\boldsymbol{b}\,\boldsymbol{c}. \tag{8.4}$$

We thus have, with (6.8),

$$\operatorname{tr}\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{c}^2 = \tfrac{1}{2}\operatorname{tr}\boldsymbol{a}(\boldsymbol{c}\,\boldsymbol{b}\,\boldsymbol{c}^2 + \boldsymbol{c}^2\boldsymbol{b}\,\boldsymbol{c}) \equiv 0.$$

Thus, an integrity basis for the three symmetric 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, under the orthogonal group is given by the integrity bases for the three pairs of matrices which can be selected from $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, the invariants

$$\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}^2\boldsymbol{c}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}^2 \quad \text{and} \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}, \tag{8.5}$$

and the invariants which can be formed from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$.

Four matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$. In a similar manner to that employed above, we see that an integrity basis for the four symmetric 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ under the orthogonal group is given by the integrity bases for the four sets of three matrices which can be selected from $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, the invariants

$$\operatorname{tr}\boldsymbol{b}\,\boldsymbol{a}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{a}^2, \quad \operatorname{tr}\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{a}\,\boldsymbol{d}\,\boldsymbol{a}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}^2\boldsymbol{d}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}, \tag{8.6}$$

and the invariants obtained from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$.

Five matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$. Again, we see from Theorem 3, that an integrity basis for the five symmetric 3×3 matrices $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$ under the orthogonal group is given by the integrity bases for the five sets of four matrices which can be selected from $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$, the invariants

$$\begin{aligned}&\operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}^2\boldsymbol{e}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{e}\,\boldsymbol{d}\,\boldsymbol{e}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{e}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e}^2,\\ &\qquad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{e}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e}^2, \quad \operatorname{tr}\boldsymbol{a}\,\boldsymbol{b}\,\boldsymbol{c}\,\boldsymbol{d}\,\boldsymbol{e}\end{aligned} \tag{8.7}$$

and the invariants obtained from these by permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$, $\boldsymbol{e}$.

The integrity bases given in this section are redundant, since, except in the case of the integrity basis for a single matrix, certain of their elements can be expressed as polynomials in the others. Less highly redundant integrity bases for five or fewer matrices will be derived in a later paper. However, the integrity bases given here have the possible merit of symmetry with respect to interchange of the matrices.

9. Application of Peano's theorem to the determination of an integrity basis for $\boldsymbol{R}$ symmetric 3×3 matrices

Let us define the elements of the R symmetric 3×3 matrices $\boldsymbol{a}_P$ $(P=1, 2, \dots, R)$ by

$$\boldsymbol{a}_P = \|a_{ij}^{(P)}\|. \tag{9.1}$$

Let us suppose that $I_1, I_2, \dots, I_\mu$ is a set of polynomial invariants which forms an integrity basis for the five matrices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5$ under the full or proper orthogonal group. Then, it follows from PEANO's theorem (see, for example, [3]) that an integrity basis for the R matrices $\boldsymbol{a}_P$ $(P=1, 2, \dots, R)$ may be obtained by polarization of $I_1, I_2, \dots, I_\mu$ and of the invariant J of the six matrices $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5, \boldsymbol{a}_6$ defined by

$$J = \begin{vmatrix} a_{11}^{(1)}, & a_{22}^{(1)}, & a_{33}^{(1)}, & a_{23}^{(1)}, & a_{31}^{(1)}, & a_{12}^{(1)} \\ a_{11}^{(2)}, & a_{22}^{(2)}, & a_{33}^{(2)}, & a_{23}^{(2)}, & a_{31}^{(2)}, & a_{12}^{(2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{11}^{(6)}, & a_{22}^{(6)}, & a_{33}^{(6)}, & a_{23}^{(6)}, & a_{31}^{(6)}, & a_{12}^{(6)} \end{vmatrix}. \tag{9.2}$$

Thus, an integrity basis for the R matrices is given by

$$a_{i_1 j_1}^{(K_1)} a_{i_2 j_2}^{(K_2)} \dots a_{i_\chi j_\chi}^{(K_\chi)} \frac{\partial^\chi I_\alpha}{\partial a_{i_1 j_1}^{(L_1)} \partial a_{i_2 j_2}^{(L_2)} \dots \partial a_{i_\chi j_\chi}^{(L_\chi)}}$$

and

$$a_{i_1 j_1}^{(K_1)} a_{i_2 j_2}^{(K_2)} \dots a_{i_\chi j_\chi}^{(K_\chi)} \frac{\partial^\chi J}{\partial a_{i_1 j_1}^{(L_1)} \partial a_{i_2 j_2}^{(L_2)} \dots \partial a_{i_\chi j_\chi}^{(L_\chi)}}, \tag{9.3}$$

where $\alpha = 1, 2, \dots, \mu$; $K_1, K_2, \dots, K_\chi, L_1, L_2, \dots, L_\chi$ are integers, not necessarily all different, chosen from $1, 2, \dots, R$ and χ is any positive integer.

We note that the invariants generated in (9.3) from I_α and J have the same total degrees as I_α and J respectively. Now, taking $R=5$ in Theorem 3, we obtain an integrity basis for five symmetric 3×3 matrices, the members of which have total degrees $\leqq 7$. This is given in § 8. Also, from (9.2), J has total degree 6. Thus, if we take as $I_1, I_2, \dots, I_\mu$ the members of the integrity basis for the five matrices $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_5$ given in § 8, equations (9.3) generate invariants of total degree $\leqq 7$.

Since any invariant of the R symmetric matrices $\boldsymbol{a}_P$ of total degree ν is expressible as a polynomial in traces of matrix products formed from $\boldsymbol{a}_P$ of total degree $\leqq \nu$, an invariant of the R matrices $\boldsymbol{a}_P$ of total degree $\leqq 7$ must be expressible as a polynomial in traces of matrix products which either

(i) involve five or fewer matrices,

or (ii) have total degree 7, involve six matrices and are of partial degree 1 in each of five matrices and of partial degree 2 in the sixth matrix,

or *(iii)* have total degree 7, involve seven matrices and are of partial degree 1 in each of them,

or *(iv)* have total degree 6, involve six matrices and are of partial degree 1 in each of them.

With Theorem 3, we obtain

Theorem 4. *An integrity basis for R symmetric 3×3 matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$, under the full or the proper orthogonal group, is formed by the integrity bases for each set of five matrices which can be selected from $\boldsymbol{a}_P$, together with the invariants*

$$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6},$$

$$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6}^2$$

and

$$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6} \boldsymbol{a}_{K_7},$$

where $K_1, K_2, \ldots, K_7$ are integers, all different, selected from 1, 2, ..., R.

10. Application of Peano's theorem to the contraction of isotropic matrix polynomials

Let $\boldsymbol{P}$ be a symmetric matrix polynomial in the R symmetric 3×3 matrices $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ in which the coefficients of the matrix products are either numerical constants or polynomial invariants under the orthogonal group of the matrices $\boldsymbol{a}_P$. $\boldsymbol{P}$ is then, of course, an isotropic matrix polynomial. Let $\boldsymbol{y}$ be an arbitrary symmetric 3×3 matrix. Let $I_1, I_2, \ldots, I_\mu$ be a set of polynomial invariants which form an integrity basis for the R matrices $\boldsymbol{a}_P$ under the orthogonal group. There exists an integrity basis for the R matrices $\boldsymbol{a}_P$ and the matrix $\boldsymbol{y}$ which consists of $I_1, I_2, \ldots, I_\mu$ together with additional invariants each of which involves $\boldsymbol{y}$. Let $J_1, J_2, \ldots, J_\mu$ be those additional invariants in the integrity basis for $\boldsymbol{a}_P$ $(P=1, 2, \ldots, R)$ any $\boldsymbol{y}$ which are linear in the elements of $\boldsymbol{y}$.

Since $\operatorname{tr} \boldsymbol{y}\boldsymbol{P}$ is a scalar invariant under the orthogonal group (full or proper) which is linear in the elements of $\boldsymbol{y}$, it can be expressed in the form

$$\operatorname{tr} \boldsymbol{y}\boldsymbol{p} = \sum_{\alpha=1}^{\eta} \varphi_\alpha J_\alpha, \tag{10.1}$$

where φ_α $(\alpha=1, 2, \ldots, \eta)$ are polynomials in $I_1, I_2, \ldots, I_\mu$. If we define the elements of $\boldsymbol{P}$ by

$$\boldsymbol{P} = \|P_{ij}\|,$$

then, from (10.1), we have

$$P_{ij} = \frac{1}{2}\left[\frac{\partial}{\partial y_{ij}}(\operatorname{tr} \boldsymbol{y}\boldsymbol{P}) + \frac{\partial}{\partial y_{ji}}(\operatorname{tr} \boldsymbol{y}\boldsymbol{P})\right] = \frac{1}{2}\sum_{\alpha=1}^{\eta} \varphi_\alpha\left(\frac{\partial J_\alpha}{\partial y_{ij}} + \frac{\partial J_\alpha}{\partial y_{ji}}\right). \tag{10.2}$$

Now, suppose we choose the integrity basis for $\boldsymbol{y}$ and the R matrices $\boldsymbol{a}_P$ in accordance with Theorem 4. Then, each of the invariants J_α is either

(i) linear in $\boldsymbol{y}$ and involves four or fewer of the matrices $\boldsymbol{a}_P$;

or *(ii)* has the form $\operatorname{tr} \boldsymbol{y}\boldsymbol{a}_{K_1}\boldsymbol{a}_{K_2}\boldsymbol{a}_{K_3}\boldsymbol{a}_{K_4}\boldsymbol{a}_{K_5}$, in which case

$$\left\|\frac{\partial J_\alpha}{\partial y_{ij}} + \frac{\partial J_\alpha}{\partial y_{ji}}\right\| = \boldsymbol{a}_{K_1}\boldsymbol{a}_{K_2}\boldsymbol{a}_{K_3}\boldsymbol{a}_{K_4}\boldsymbol{a}_{K_5} + \boldsymbol{a}_{K_5}\boldsymbol{a}_{K_4}\boldsymbol{a}_{K_3}\boldsymbol{a}_{K_2}\boldsymbol{a}_{K_1};$$

or *(iii)* has one of the forms $\operatorname{tr} \boldsymbol{y} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5}$,

$$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{y} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5}, \quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{y} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5},$$
$$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{y} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} \quad \text{or} \quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{y} \boldsymbol{a}^2_{K_5},$$

in which cases,

$$\left\| \frac{\partial J_\alpha}{\partial y_{ij}} + \frac{\partial J_\alpha}{\partial y_{ji}} \right\| = \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} + \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1},$$
$$\boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_1} + \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2},$$
$$\boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} + \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3},$$
$$\boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} + \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4},$$

or

$$\boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5},$$

respectively; or

(iv) has the form $\operatorname{tr} \boldsymbol{y} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6}$, in which case

$$\left\| \frac{\partial J_\alpha}{\partial y_{ij}} + \frac{\partial J_\alpha}{\partial y_{ji}} \right\| = \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6} + \boldsymbol{a}_{K_6} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1}.$$

In cases *(ii)* or *(iii)*, K_1, K_2, K_3, K_4, K_5 are five numbers, all different, of the set $1, 2, \ldots, R$; in case *(iv)*, $K_1, K_2, \ldots, K_6$ are six numbers, all different, of the set $1, 2, \ldots, R$. We note that in case *(iii)*, when we take into account the fact that all selections of $K_1, K_2, \ldots, K_5$ (all different) from $1, 2, \ldots, R$ are possible, only the first three of the forms given for $\|\partial J_\alpha/\partial y_{ij} + \partial J_\alpha/\partial y_{ji}\|$ are essentially different.

In case *(i)*, $\partial J_\alpha/\partial y_{ij} + \partial J_\alpha/\partial y_{ji}$ is a symmetric matrix polynomial in four of the matrices $\boldsymbol{a}_P$, say, $\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \boldsymbol{a}_{K_3}, \boldsymbol{a}_{K_4}$. From § 4, we see that such a matrix polynomial may be expressed as a matrix polynomial in which the terms have the forms (4.7) and (4.10) with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ replaced by $\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \boldsymbol{a}_{K_3}, \boldsymbol{a}_{K_4}$.

Combining the results for cases (i), (ii), (iii) and *(iv)* we obtain

Theorem 5. *Any symmetric isotropic matrix polynomial in R symmetric 3×3 matrices $\boldsymbol{a}_P$ $(P = 1, 2, \ldots, R)$ may be expressed as an isotropic matrix polynomial in which the terms have the forms*

$$\boldsymbol{I}, \quad \boldsymbol{a}_{K_1}, \quad \boldsymbol{a}^2_{K_1};$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} + \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1}, \quad \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_2} + \boldsymbol{a}_{K_2} \boldsymbol{a}^2_{K_1}, \quad \boldsymbol{a}^2_{K_1} \boldsymbol{a}^2_{K_2} + \boldsymbol{a}^2_{K_2} \boldsymbol{a}^2_{K_1};$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} + \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1}, \quad \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} + \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}^2_{K_1},$$
$$\boldsymbol{a}_{K_2} \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_3} + \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_2}, \quad \boldsymbol{a}^2_{K_1} \boldsymbol{a}^2_{K_2} \boldsymbol{a}_{K_3} + \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_2} \boldsymbol{a}^2_{K_1},$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_3} + \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1}, \quad \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_1} + \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1};$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1},$$
$$\boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}^2_{K_1}, \quad \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_2} \boldsymbol{a}_{K_1},$$
$$\boldsymbol{a}^2_{K_1} \boldsymbol{a}^2_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_2} \boldsymbol{a}^2_{K_1}, \quad \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_2} \boldsymbol{a}^2_{K_3} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_3} \boldsymbol{a}^2_{K_2} \boldsymbol{a}_{K_1},$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_1} + \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1}, \quad \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_1} + \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2},$$
$$\boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_1} + \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2}, \quad \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_3} \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_4} + \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_1} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2};$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} + \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1}, \quad \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} + \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1},$$
$$\boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_1} + \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2}, \quad \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} + \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1} \boldsymbol{a}^2_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3},$$
$$\boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6} + \boldsymbol{a}_{K_6} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_1};$$

for all selections of $K_1, K_2, \ldots, K_6$ (all different) chosen from $1, 2, \ldots, R$, and the coefficients are polynomial scalar invariants under the orthogonal group of $\boldsymbol{a}_P$ $(P = 1, 2, \ldots, R)$.

We may also express the symmetric isotropic matrix polynomial $\boldsymbol{P}$ as a symmetric isotropic matrix polynomial, in which each of the matrix terms involves no more than six matrices, from a knowledge of the elements of an integrity basis for seven symmetric matrices which are linear in one of them. Let $I_\alpha(\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \ldots, \boldsymbol{a}_{K_6}, \boldsymbol{y})$ be the elements of an integrity basis for the seven symmetric matrices $\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \ldots, \boldsymbol{a}_{K_6}$ and $\boldsymbol{y}$, which are linear in the elements of $\boldsymbol{y}$. Then, $\operatorname{tr} \boldsymbol{y}\boldsymbol{p}$ may be expressed in the form

$$\operatorname{tr} \boldsymbol{y}\boldsymbol{p} = \sum \sum_{\alpha} \psi_\alpha I_\alpha(\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \ldots, \boldsymbol{a}_{K_6}, \boldsymbol{y}), \tag{10.3}$$

where the coefficients ψ_α are polynomials in the elements of an integrity basis for the R matrices $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_R$, $\sum_{\alpha}$ denotes summation over the invariants $I_\alpha(\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \ldots, \boldsymbol{a}_{K_6}, \boldsymbol{y})$ and $\sum$ denotes summation over all possible selections of the six matrices $\boldsymbol{a}_{K_1}, \boldsymbol{a}_{K_2}, \ldots, \boldsymbol{a}_{K_6}$ from $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_R$. From (10.3), we obtain

$$P_{ij} = \frac{1}{2} \sum \sum_{\alpha} \psi_\alpha \left(\frac{\partial I_\alpha}{\partial y_{ij}} + \frac{\partial I_\alpha}{\partial y_{ji}} \right). \tag{10.4}$$

The form given by (10.4) for $\boldsymbol{P} (= \|P_{ij}\|)$ will depend, in general, on the integrity basis chosen for I_α.

11. Appendix

In § 2 we remarked that the relation (2.2) which is satisfied by any three 3×3 matrices may be derived from the Hamilton-Cayley theorem. If the matrices $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ in (2.2) are defined by

$$\boldsymbol{a} = \|a_{ij}\|, \quad \boldsymbol{b} = \|b_{ij}\| \quad \text{and} \quad \boldsymbol{c} = \|c_{ij}\|, \tag{11.1}$$

then it can be easily shown that the relation (2.2) is equivalent to the relation

$$\begin{vmatrix} \delta_{i_1 j_1}, & \delta_{i_1 j_2}, & \delta_{i_1 j_3}, & \delta_{i_1 j_4} \\ \delta_{i_2 j_1}, & \delta_{i_2 j_2}, & \delta_{i_2 j_3}, & \delta_{i_2 j_4} \\ \delta_{i_3 j_1}, & \delta_{i_3 j_2}, & \delta_{i_3 j_3}, & \delta_{i_3 j_4} \\ \delta_{i_4 j_1}, & \delta_{i_4 j_2}, & \delta_{i_4 j_3}, & \delta_{i_4 j_4} \end{vmatrix} a_{i_1 j_1} b_{i_2 j_2} c_{i_3 j_3} = 0, \tag{11.2}$$

by expanding the determinant and interpreting each of the terms obtained in matrix notation.

By a similar procedure, it can be shown that the Hamilton-Cayley theorem for a 3×3 matrix $\boldsymbol{a}$ is equivalent to the relation

$$\begin{vmatrix} \delta_{i_1 j_1}, & \delta_{i_1 j_2}, & \delta_{i_1 j_3}, & \delta_{i_1 j_4} \\ \delta_{i_2 j_1}, & \delta_{i_2 j_2}, & \delta_{i_2 j_3}, & \delta_{i_2 j_4} \\ \delta_{i_3 j_1}, & \delta_{i_3 j_2}, & \delta_{i_3 j_3}, & \delta_{i_3 j_4} \\ \delta_{i_4 j_1}, & \delta_{i_4 j_2}, & \delta_{i_4 j_3}, & \delta_{i_4 j_4} \end{vmatrix} a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} = 0. \tag{11.3}$$

Both (11.2) and (11.3) follow immediately from the evident validity of the relation

$$\begin{vmatrix} \delta_{i_1 j_1}, & \delta_{i_1 j_2}, & \delta_{i_1 j_3}, & \delta_{i_1 j_4} \\ \delta_{i_2 j_1}, & \delta_{i_2 j_2}, & \delta_{i_2 j_3}, & \delta_{i_2 j_4} \\ \delta_{i_3 j_1}, & \delta_{i_3 j_2}, & \delta_{i_3 j_3}, & \delta_{i_3 j_4} \\ \delta_{i_4 j_1}, & \delta_{i_4 j_2}, & \delta_{i_4 j_3}, & \delta_{i_4 j_4} \end{vmatrix} = 0. \tag{11.4}$$

The relation (11.2) can be derived from (11.3) by a polarization procedure. Denoting the expression on the left-hand side of (11.2) by $F_{i_4 j_4}$ and that on the left-hand side of (11.3) by $G_{i_4 j_4}$, it is easily seen that

$$6 F_{i_4 j_4} = b_{kl} \, c_{mn} \, \frac{\partial G_{i_4 j_4}}{\partial a_{kl} \, \partial a_{mn}}, \tag{11.5}$$

and hence that (11.2) follows from (11.3).

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Einstein's Equations and Classical Hydrodynamics

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1. Introduction

As is well-known, the theory of general relativity involves field equations, known as Einstein's equations, which equate the energy-tensor that describes the physical properties of matter to the geometrical Einstein tensor. A constant of proportionality is involved in these equations which has to be evaluated by an approximation procedure that reduces Einstein's equations to those of Newtonian gravitational theory. Traditionally this has always been done by working out the Einstein tensor for a statical, orthogonal and isotropic metric whose metrical tensor departs slightly from that of special relativity (MCVITTIE 1956a). It is then shown that Poisson's equation can be obtained by choosing the constant of proportionality to be itself proportional to the constant of gravitation.

The question arises: What equations of classical theory would be obtained if one starts with metrics whose metrical tensors have more complicated forms than the simple type traditionally assumed? A partial answer to this question has been provided by one of us (MCVITTIE 1956b). Assuming an orthogonal metrical tensor of a somewhat more general form than that commonly used and an energy tensor appropriate to a perfect fluid, the equations of motion and of continuity of classical hydrodynamics were arrived at. Moreover the procedure led to solutions of these classical equations in terms of indeterminate functions, one of which was shown to be the gravitational potential for the self-attraction of the fluid. Incidentally, it may be remarked that it is possible to solve the equations of classical hydrodynamics in a very general manner in terms of indeterminate functions directly and without the use of general relativity (FINZI 1934; WHITHAM 1954) though the physical interpretation of the resulting functions is not entirely clear.

In MCVITTIE's treatment of the reduction of Einstein's equations to their classical counterparts two points emerged: firstly, it was evident that two approximation parameters were involved, one of which was essentially the constant of gravitation, while the other contained the reciprocal of the square of the velocity of light. Secondly, the solutions of the equations of hydrodynamics so obtained had the following property: if ϱ is the density of the fluid, and $\boldsymbol{U}$ the fluid-velocity, then the momentum-vorticity, $\nabla \times (\varrho \boldsymbol{U})$, was necessarily zero. It was conjectured that this restriction arose because of the initial use of an orthogonal metrical tensor.

The aims of the present investigation are, firstly, to put the approximation procedure on a sound basis by the introduction of dimensionless variables in the general relativity equations. It is shown that there are two approximation

parameters, denoted by η and ε (Eqn. (2.06) below) the first of which involves the constant of gravitation, the second the ratio of a standard velocity to the velocity of light. Secondly, ways of removing the restriction to fluid motions implied by the vanishing of the momentum-vorticity are examined. It is shown that starting with a non-orthogonal metric certainly removes the vanishing of the momentum-vorticity. But, by a suitable application of infinitesimal co-ordinate-transformations, it can be proved that certain kinds of *orthogonal* metrical tensors (Eqn. (4.38)) can also lead to a non-vanishing momentum-vorticity. The nature of these metrical tensors may be seen by comparing (4.37) and (4.38) with (3.04). In the latter case the terms which do not have ε^2 as a factor involve one function only, the gravitational potential ψ. But in the former case, the corresponding terms are all different. Thus a zero momentum-vorticity is not associated with the orthogonality of the metrical tensor but with the detailed form of the components of this tensor.

In Section 5 we return to the nature of the approximation procedure. EINSTEIN & INFELD (1949) have proposed a method of successive approximations in dealing with Einstein's equations which employs a single arbitrary small parameter λ. The relationship of this parameter to η and ε is examined and it is shown that it corresponds to ε and cannot be identified with η. Thus the degree of arbitrariness of λ appears to be limited.

2. Dimensionless variables

In order that the magnitudes of variables and functions with physical dimensions may be compared, dimensionless quantities will be introduced. They will be related to ordinary physical variables and functions expressed in cgs units in order to facilitate any future physical applications.

Consider the following metric in the cgs system

$$ds^2 = (1 + \varkappa V_{44})(dx^4)^2 - \frac{2}{c}\sum_{i=1}^{3} \varkappa V_{4i}\, dx^i\, dx^4 - \frac{1}{c^2}\sum_{i=1}^{3}(1 + \varkappa V_{ii})(dx^i)^2, \quad (2.01)$$

where c is the velocity of light, $\varkappa = 8\pi G/c^2$ and G is the gravitational constant. The physical dimensions of the above quantities are,

s		time,
x^4		time,
x^i	$(i = 1, 2, 3)$	length,
c		$\frac{\text{length}}{\text{time}}$,
$\varkappa$		$\frac{\text{length}}{\text{mass}}$,
$V_{\mu\nu}$	$(\mu, \nu = 4, 1, 2, 3)$	$\frac{\text{mass}}{\text{length}}$.

Thus the $V_{\mu\nu}$ have the same dimensions as gravitational potentials. If an arbitrary fixed length, l, and an arbitrary fixed time-interval, t, are chosen, dimension-

less coordinates, ξ^μ, may be defined as follows:

$$\begin{aligned} \xi^i &= \frac{x^i}{l} \qquad (i = 1, 2, 3), \\ \xi^4 &= \frac{x^4}{t}. \end{aligned} \tag{2.02}$$

The equations (2.02) define a coordinate transformation under which the metric (2.01) becomes

$$d s^2 = t^2 (1 + \varkappa V_{44}) (d\xi^4)^2 - 2 \frac{l\,t}{c} \sum_{i=1}^{3} \varkappa V_{4i} \, d\xi^i \, d\xi^4 - \frac{l^2}{c^2} \sum_{i=1}^{3} (1 + \varkappa V_{ii}) (d\xi^i)^2. \tag{2.03}$$

Dividing (2.03) by t^2 and defining $\sigma = s/t$ yields

$$d\sigma^2 = (1 + \varkappa V_{44}) (d\xi^4)^2 - 2 \frac{l/t}{c} \sum_{i=1}^{3} \varkappa V_{4i} \, d\xi^i \, d\xi^4 - \frac{(l/t)^2}{c^2} \sum_{i=1}^{3} (1 + \varkappa V_{ii}) (d\xi^i)^2. \tag{2.04}$$

The expressions $\varkappa V_{\mu\nu}$ are dimensionless, but $\varkappa$ and $V_{\mu\nu}$ separately are not. Let V be an arbitrary fixed gravitational potential, and let $\varkappa V_{\mu\nu} = \eta \gamma_{\mu\nu}$, where $\eta = \varkappa V$ and $\gamma_{\mu\nu} = V_{\mu\nu}/V$. Further, let $v = l/t$ and $\varepsilon = v/c$, then (2.04) becomes

$$d\sigma^2 = (1 + \eta \gamma_{44}) (d\xi^4)^2 - 2 \varepsilon \sum_{i=1}^{3} \eta \gamma_{4i} \, d\xi^i \, d\xi^4 - \varepsilon^2 \sum_{i=1}^{3} (1 + \eta \gamma_{ii}) (d\xi^i)^2, \tag{2.05}$$

where the dimensionless variables are related to those in (2.01) by

$$\begin{aligned} \xi^i &= \frac{x^i}{l} \qquad (i = 1, 2, 3), \\ \xi^4 &= \frac{x^4}{t}, \\ \sigma &= \frac{s}{t}, \\ \eta &= \varkappa V = \frac{8 \pi G V}{c^2}, \\ \gamma_{\mu\nu} &= \frac{V_{\mu\nu}}{V}, \\ \varepsilon &= \frac{v}{c}, \\ v &= \frac{l}{t}. \end{aligned} \tag{2.06}$$

The metrical tensor $g_{\mu\nu}$ for the metric (2.05) is

$$\begin{aligned} g_{44} &= 1 + \eta \gamma_{44}, \\ g_{4i} &= -\varepsilon \eta \gamma_{4i} \qquad && (i = 1, 2, 3), \\ g_{ii} &= -\varepsilon^2 (1 + \eta \gamma_{ii}) \qquad && (i = 1, 2, 3), \\ g_{ij} &= 0 && (i \neq j). \end{aligned} \tag{2.07}$$

The energy tensor and Einstein's equations are known for the metric (2.01) in cgs units (McVittie 1956c). Let $\bar{g}_{\mu\nu}$, $\bar{R}_{\mu\nu}$, and $\bar{T}^{\mu\nu}$ be the metrical, Ricci, and

energy tensors corresponding to the metric (2.01). Then

$$d s^2 = \bar{g}_{\mu\nu}\, d x^\mu\, d x^\nu, \tag{2.08}$$

$$\bar{R}_{\mu\nu} = \frac{\partial^2 \ln \sqrt{-\bar{g}}}{\partial x^\mu\, \partial x^\nu} - \frac{\partial}{\partial x^\sigma} \left\{ {\sigma \atop \mu\nu} \right\} + \left\{ {\tau \atop \mu\sigma} \right\} \left\{ {\sigma \atop \nu\tau} \right\} - \left\{ {\tau \atop \mu\nu} \right\} \frac{\partial \ln \sqrt{-\bar{g}}}{\partial x^\tau}, \tag{2.09}$$

$$\bar{T}^{\mu\nu} = \left(\bar{\varrho} + \frac{\bar{p}}{c^2}\right) \frac{d x^\mu}{d s} \frac{d x^\nu}{d s} - \bar{g}^{\mu\nu} \frac{\bar{p}}{c^2}, \tag{2.10}$$

$$-8\pi G \bar{T}^{\mu\nu} = \bar{R}^{\mu\nu} - \tfrac{1}{2} \bar{g}^{\mu\nu} \bar{R}, \tag{2.11}$$

where the Christoffel symbols in (2.09) are defined in terms of (2.08), and in (2.10) $\bar{\varrho}$ and $\bar{p}$ are the density and pressure in cgs units. In order to find expressions corresponding to (2.09) to (2.11) for the metric (2.05), the transformation (2.02) will be used as an intermediary step. This transformation will be denoted by a prime. Let $\bar{g}'_{\mu\nu}$, $\bar{R}'_{\mu\nu}$, and $\bar{T}'^{\mu\nu}$ be the transforms of $\bar{g}_{\mu\nu}$, $\bar{R}_{\mu\nu}$, and $\bar{T}^{\mu\nu}$ under the transformation (2.02). Then, since $\bar{g}_{\mu\nu}$ is given by (2.01), $\bar{g}'_{\mu\nu}$ by (2.03), and $g_{\mu\nu}$ by (2.05), it is easily seen that

$$\begin{aligned} g_{\mu\nu} &= \frac{1}{t^2} \bar{g}'_{\mu\nu}, \\ g^{\mu\nu} &= t^2 \bar{g}'^{\mu\nu}. \end{aligned} \tag{2.12}$$

The Ricci tensor $R_{\mu\nu}$ for the metric (2.05) is defined by analogy with (2.09) as follows:

$$R_{\mu\nu} = \frac{\partial^2 \ln \sqrt{-g}}{\partial \xi^\mu\, \partial \xi^\nu} - \frac{\partial}{\partial \xi^\sigma} \left\{ {\sigma \atop \mu\nu} \right\} + \left\{ {\sigma \atop \mu\tau} \right\} \left\{ {\tau \atop \nu\sigma} \right\} - \left\{ {\tau \atop \mu\nu} \right\} \frac{\partial \ln \sqrt{-g}}{\partial \xi^\tau}, \tag{2.13}$$

where now the Christoffel symbols are defined in terms of the metric (2.05). Combining (2.13) and (2.12) with (2.09), gives

$$\begin{aligned} R_{\mu\nu} &= \bar{R}'_{\mu\nu}, \\ R^{\mu\nu} &= t^4 \bar{R}'^{\mu\nu}. \end{aligned} \tag{2.14}$$

The energy tensor $T^{\mu\nu}$ for the dimensionless metric (2.05) should also be dimensionless. By analogy with (2.10), $T^{\mu\nu}$ is defined as follows

$$T^{\mu\nu} = (\varrho + \varepsilon^2 p) \left(\frac{d\xi^\mu}{d\sigma}\right) \left(\frac{d\xi^\nu}{d\sigma}\right) - g^{\mu\nu} \varepsilon^2 p, \tag{2.15}$$

where

$$\begin{aligned} \varrho &= \bar{\varrho} \frac{l^2}{V}, \\ p &= \bar{p} \frac{t^2}{V}. \end{aligned} \tag{2.16}$$

By comparing (2.15) with (2.10), $T^{\mu\nu}$ can be found in terms of $\bar{T}'^{\mu\nu}$. Thus

$$\begin{aligned} T^{\mu\nu} &= (\varrho + \varepsilon^2 p) \left(\frac{d\xi^\mu}{d\sigma}\right) \left(\frac{d\xi^\nu}{d\sigma}\right) - g^{\mu\nu} \varepsilon^2 p \\ &= \frac{t^2 l^2}{V} \left[\left(\bar{\varrho} + \frac{\bar{p}}{c^2}\right) \left(\frac{d\xi^\mu}{ds}\right) \left(\frac{d\xi^\nu}{ds}\right) - \bar{g}'^{\mu\nu} \frac{\bar{p}}{c^2} \right] \\ &= \frac{t^2 l^2}{V} \bar{T}'^{\mu\nu}. \end{aligned} \tag{2.17}$$

Einstein's equations can now be derived for the dimensionless metric (2.05). Since (2.11) is a tensor equation, it remains valid after a coordinate transformation. Thus

$$-8\pi G\overline{T}'^{\mu\nu}=\overline{R}'^{\mu\nu}-\tfrac{1}{2}\overline{g}'^{\mu\nu}\overline{R}'. \tag{2.18}$$

Substituting the dimensionless tensors given by (2.12), (2.14), and (2.17) into (2.18) gives

$$-\frac{8\pi G V}{l^2 t^2}T^{\mu\nu}=\frac{1}{t^4}R^{\mu\nu}-\frac{1}{2}\cdot\frac{1}{t^4}g^{\mu\nu}R,$$

or

$$-\frac{8\pi G V}{v^2}T^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R, \tag{2.19}$$

or

$$-\frac{\eta}{\varepsilon^2}T^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R.$$

The Newtonian approximation to Einstein's equations (2.19) will be examined in the course of the following sections. That approximation will be defined as the result of first cancelling factors $-\eta/\varepsilon^2$ on both sides of Einstein's equations and then setting ε^2 equal to zero. The second step implies that the standard velocity v is such that its square is negligibly small compared with the square of the velocity of light. Let

$$u^\mu=\frac{d\xi^\mu}{d\sigma}; \tag{2.20}$$

then, neglecting terms of order η, (2.05) becomes

$$1=(u^4)^2-\varepsilon^2\sum_{i=1}^{3}(u^i)^2, \tag{2.21}$$

so that in the Newtonian approximation u^4 becomes equal to unity. The u^i $(i=1,2,3)$ become the dimensionless Newtonian velocity components U_i where the $v\,U_i$ are the ordinary Newtonian velocity components expressed in cgs units. The density ϱ and the pressure p become the dimensionless Newtonian density and pressure where $\varrho V/l^2$ and $p V/t^2$ are the ordinary Newtonian density and pressure. The coordinate ξ^4 becomes T where $t\,T$ is the Newtonian absolute time in seconds, and the ξ^i $(i=1,2,3)$ become X_i where the lX_i are the usual Newtonian space coordinates. The $\gamma_{\mu\nu}$ become the same functions of T and the X_i as they were of the ξ^μ $(\mu=4,1,2,3)$.

In connection with the above symbols, certain conventions will be employed. Greek indices will range over the values 4, 1, 2, 3 and Latin indices will take on the values 1, 2, 3 only. The indices l, m, n will always denote a cyclic permutation of 1, 2, 3. A comma will be used to denote partial differentiation with respect to the coordinate indicated by the subscript after the comma. For example,

$$\gamma_{33,14}\equiv\frac{\partial^2\gamma_{33}}{\partial\xi^1\,\partial\xi^4}.$$

However, the same symbol $\gamma_{33,14}$ will be used to denote the derivative with respect to the Newtonian coordinates when no ambiguity can arise. In such an instance

$$\gamma_{33,14}\equiv\frac{\partial^2\gamma_{33}}{\partial X_1\,\partial T}.$$

The symbol ∇^2 will be similarly used in both contexts. Thus

$$\nabla^2 = \frac{\partial^2}{(\partial \xi^1)^2} + \frac{\partial^2}{(\partial \xi^2)^2} + \frac{\partial^2}{(\partial \xi^3)^2},$$

or

$$\nabla^2 = \frac{\partial^2}{(\partial X_1)^2} + \frac{\partial^2}{(\partial X_2)^2} + \frac{\partial^2}{(\partial X_3)^2},$$

whichever is applicable.

3. Approximate forms of Einstein's equations. Non-orthogonal metrics. Hydrodynamics

The elements $g_{\mu\nu}$ of the metrical tensor have already been expanded in powers of η in equations (2.07), with η^2 and higher powers of η being neglected. Further, the $\gamma_{\mu\nu}$ of (2.07) may be expanded in powers of ε. The simplest case is

$$\gamma_{44} = -\psi,$$
$$\gamma_{ll} = \psi,$$
$$\gamma_{4l} = 0,$$

or

$$d\sigma^2 = (1 - \eta\psi)(d\xi^4)^2 - \varepsilon^2 \sum_{i=1}^{3} (1 + \eta\psi)(d\xi^i)^2. \tag{3.01}$$

Again neglecting powers of η greater than one, Einstein's equations as given by DINGLE's formulae (McVITTIE 1956d) are

$$\begin{aligned} -\frac{\eta}{\varepsilon^2} T^{44} &= \frac{\eta}{\varepsilon^2} \nabla^2 \psi, \\ -\frac{\eta}{\varepsilon^2} T^{4l} &= -\frac{\eta}{\varepsilon^2} \psi_{,4l}, \\ -\frac{\eta}{\varepsilon^2} T^{lm} &= 0, \\ -\frac{\eta}{\varepsilon^2} T^{ll} &= \frac{\eta}{\varepsilon^2} \psi_{,44}, \end{aligned} \tag{3.02}$$

where $T^{\mu\nu}$ is given by equation (2.15). Following the procedure outlined in Section 2, the Newtonian approximation to equations (3.02) is

$$\begin{aligned} \varrho &= -\nabla^2 \psi, \\ \varrho U_l &= \psi_{,4l}, \\ \varrho U_l U_m &= 0, \\ \varrho U_l^2 + p &= -\psi_{,44}. \end{aligned} \tag{3.03}$$

Clearly, in the above solution to the equations of hydrodynamics, at most one of the velocity components can be non-zero. Thus the very simple metric (3.01) leads to a very restricted solution.

Since it is more convenient to work with orthogonal metrics, the next step in expanding the $\gamma_{\mu\nu}$ in powers of ε could be

$$\gamma_{44} = -(\psi + 2\varepsilon^2 \psi_4),$$
$$\gamma_{ll} = \psi + 2\varepsilon^2 \psi_l,$$
$$\gamma_{4l} = 0,$$

or

$$d\sigma^2 = [1 - \eta(\psi + 2\varepsilon^2\psi_4)]\,(d\xi^4)^2 - \varepsilon^2 \sum_{i=1}^{3} [1 + \eta(\psi + 2\varepsilon^2\psi_i)]\,(d\xi^i)^2. \quad (3.04)$$

McVittie (1956b) has worked out the consequences of (3.04) but with $\psi_4 = 0$. The inclusion of the quantity ψ_4 is made here for reasons of symmetry, and represents no essential change from McVittie's treatment. The results alone are repeated here, and the reader is referred to McVittie's work for the details. Einstein's equations for the metric (3.04) are

$$\begin{aligned}
-\frac{\eta}{\varepsilon^2} T^{44} &= \frac{\eta}{\varepsilon^2}\Big[\nabla^2\psi + \varepsilon^2 \sum_{l,m,n} (\psi_{l,mm} + \psi_{l,nn})\Big], \\
-\frac{\eta}{\varepsilon^2} T^{4l} &= -\frac{\eta}{\varepsilon^2}[\psi_{,4l} + \varepsilon^2(\psi_m + \psi_n)_{,4l}], \\
-\frac{\eta}{\varepsilon^2} T^{lm} &= \frac{\eta}{\varepsilon^2}[(\psi_n - \psi_4)_{,lm}], \\
-\frac{\eta}{\varepsilon^2} T^{ll} &= -\frac{\eta}{\varepsilon^2}[(\psi_m - \psi_4)_{,nn} + (\psi_n - \psi_4)_{,mm} - \psi_{,44} - \varepsilon^2(\psi_m + \psi_n)_{,44}],
\end{aligned} \quad (3.05)$$

with the Newtonian approximation

$$\begin{aligned}
\varrho &= -\nabla^2\psi, \\
\varrho U_l &= \psi_{,4l}, \\
\varrho U_l U_m &= -(\psi_n - \psi_4)_{,lm}, \\
\varrho U_l^2 + p &= (\psi_m - \psi_4)_{,nn} + (\psi_n - \psi_4)_{,mm} - \psi_{,44}.
\end{aligned} \quad (3.06)$$

By analogy with the definition of the vorticity of a fluid, the vector whose components are

$$\Omega_l = (\varrho U_m)_{,n} - (\varrho U_n)_{,m}, \quad (3.07)$$

will be called the *momentum-vorticity*. The motion of the fluid described by (3.06) is subject to the restriction

$$\Omega_l = 0, \quad (3.08)$$

so that its momentum-vorticity is zero. Thus, again, an orthogonal metric has led to a restricted motion.

In order to obtain a more general solution than those obtained above, a non-orthogonal metric will be used. In particular, let

$$\begin{aligned}
\gamma_{44} &= -(\psi + 2\varepsilon^2\psi_4), \\
\gamma_{ll} &= \psi + 2\varepsilon^2\psi_l, \\
\gamma_{4l} &= \varepsilon\zeta_l.
\end{aligned} \quad (3.09)$$

Since Dingle's formulae do not apply in this case, the energy tensor will be calculated from (3.09) according to (2.19). For the sake of convenience, the calculations will be carried out in terms of the functions $\gamma_{\mu\nu}$, and the identification

(3.09) will be made only at the end. From the metrical tensor

$$\begin{aligned} g_{44} &= 1 + \eta\gamma_{44}, \\ g_{ll} &= -\varepsilon^2(1+\eta\gamma_{ll}), \\ g_{4l} &= -\eta\varepsilon\gamma_{4l}, \\ g_{lm} &= 0, \end{aligned} \tag{3.10}$$

the Christoffel symbols of the second kind

$$\left\{{\alpha \atop \beta\gamma}\right\} = \tfrac{1}{2} g^{\alpha\sigma}(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma}), \tag{3.11}$$

must be computed. Since all the derivatives of the $g_{\mu\nu}$ are of order η, it is sufficient, in (3.11), to take

$$\begin{aligned} g^{44} &= 1, \\ g^{ii} &= -\frac{1}{\varepsilon^2}, \\ g^{\mu\nu} &= 0 \qquad (\mu \neq \nu). \end{aligned} \tag{3.12}$$

Thus the Christoffel symbols are found to be as follows

$$\begin{aligned} \left\{{4 \atop 44}\right\} &= \frac{\eta}{2}\gamma_{44,4}, \\ \left\{{4 \atop 4l}\right\} &= \frac{\eta}{2}\gamma_{44,l}, \\ \left\{{4 \atop ll}\right\} &= \frac{\eta\varepsilon^2}{2}\gamma_{ll,4} - \eta\varepsilon\gamma_{4l,l}, \\ \left\{{4 \atop lm}\right\} &= -\frac{\eta\varepsilon}{2}(\gamma_{4l,m} + \gamma_{4m,l}), \\ \left\{{l \atop 44}\right\} &= \frac{\eta}{\varepsilon}\gamma_{4l,4} + \frac{\eta}{2\varepsilon^2}\gamma_{44,l}, \\ \left\{{l \atop 4l}\right\} &= \frac{\eta}{2}\gamma_{ll,4}, \\ \left\{{l \atop 4m}\right\} &= \frac{\eta}{2\varepsilon}(\gamma_{4l,m} - \gamma_{4m,l}), \\ \left\{{l \atop ll}\right\} &= \frac{\eta}{2}\gamma_{ll,l}, \\ \left\{{m \atop ll}\right\} &= -\frac{\eta}{2}\gamma_{ll,m}, \\ \left\{{l \atop lm}\right\} &= \frac{\eta}{2}\gamma_{ll,m}, \\ \left\{{l \atop mn}\right\} &= 0. \end{aligned} \tag{3.13}$$

Since

$$g = -\varepsilon^6\left(1 + \eta\sum_{\lambda=1}^{4}\gamma_{\lambda\lambda}\right),$$

$$\sqrt{-g} = \varepsilon^3\left(1 + \tfrac{1}{2}\eta\sum_{\lambda=1}^{4}\gamma_{\lambda\lambda}\right),$$

then

$$\frac{\partial \ln \sqrt{-g}}{\partial \xi^\nu} = \frac{1}{2} \eta \sum_{\lambda=1}^{4} \gamma_{\lambda\lambda,\nu}. \tag{3.14}$$

In (2.13), the terms $\left\{ {\tau \atop \mu\sigma} \right\} \left\{ {\sigma \atop \nu\tau} \right\}$ and $\left\{ {\tau \atop \mu\nu} \right\} \frac{\partial \ln \sqrt{-g}}{\partial \xi^\tau}$ are, in view of (3.13) and (3.14), of order η^2 and are therefore neglected here. Using (3.13) and (3.14) in (2.13), the quantities $R_{\mu\nu}$ and R are found to be

$$\begin{aligned}
R_{44} &= \frac{\eta}{2} \sum_{i=1}^{3} \left(\gamma_{ii,44} - \frac{2}{\varepsilon} \gamma_{4i,4i} - \frac{1}{\varepsilon^2} \gamma_{44,ii} \right), \\
R_{4l} &= \frac{\eta}{2} \left[(\gamma_{nn} + \gamma_{mm})_{,4l} - \frac{1}{\varepsilon} (\gamma_{4n,ln} + \gamma_{4m,lm} - \gamma_{4l,nn} - \gamma_{4l,mm}) \right], \\
R_{lm} &= \frac{\eta}{2} \left[(\gamma_{44} + \gamma_{nn})_{,lm} + \varepsilon (\gamma_{4l,m4} + \gamma_{4m,l4}) \right], \\
R_{ll} &= \frac{\eta}{2} \left[(\gamma_{44} + \gamma_{mm} + \gamma_{nn})_{,ll} + \gamma_{ll,nn} + \gamma_{ll,mm} + 2\varepsilon \gamma_{4l,l4} - \varepsilon^2 \gamma_{ll,44} \right],
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
R = R_{44} - \frac{1}{\varepsilon^2} \sum_{l=1}^{3} R_{ll} = \eta \Bigg[&\sum_{i=1}^{3} \left(\gamma_{ii,44} - \frac{2}{\varepsilon} \gamma_{4i,4i} - \frac{1}{\varepsilon^2} \gamma_{44,ii} \right) - \\
&- \frac{1}{\varepsilon^2} \sum_{l,m,n} (\gamma_{ll,mm} + \gamma_{ll,nn}) \Bigg].
\end{aligned} \tag{3.16}$$

Finally, using (3.15), (3.16), and

$$\begin{aligned}
R^{44} &= R_{44}, \\
R^{4l} &= -\frac{1}{\varepsilon^2} R_{4l}, \\
R^{lm} &= \frac{1}{\varepsilon^4} R_{lm}, \\
R^{ll} &= \frac{1}{\varepsilon^4} R_{ll},
\end{aligned} \tag{3.17}$$

in (2.19), Einstein's equations are as follows

$$\begin{aligned}
-\frac{\eta}{\varepsilon^2} T^{44} &= \frac{\eta}{2\varepsilon^2} \sum_{l,m,n} (\gamma_{ll,mm} + \gamma_{ll,nn}), \\
-\frac{\eta}{\varepsilon^2} T^{4l} &= -\frac{\eta}{2\varepsilon^2} \Big[(\gamma_{nn} + \gamma_{mm})_{,4l} - \\
&\qquad - \frac{1}{\varepsilon} (\gamma_{4n,ln} + \gamma_{4m,lm} - \gamma_{4l,nn} - \gamma_{4l,mm}) \Big], \\
-\frac{\eta}{\varepsilon^2} T^{lm} &= \frac{\eta}{2\varepsilon^4} \left[(\gamma_{44} + \gamma_{nn})_{,lm} + \varepsilon (\gamma_{4l,m4} + \gamma_{4m,l4}) \right], \\
-\frac{\eta}{\varepsilon^2} T^{ll} &= \frac{\eta}{2\varepsilon^2} \Big[(\gamma_{mm} + \gamma_{nn})_{,44} - \frac{2}{\varepsilon} (\gamma_{4m,4m} + \gamma_{4n,4n}) - \\
&\qquad - \frac{1}{\varepsilon^2} \left((\gamma_{44} + \gamma_{nn})_{,mm} + (\gamma_{44} + \gamma_{mm})_{,nn} \right) \Big].
\end{aligned} \tag{3.18}$$

In accordance with the procedure stated in Section 2 for obtaining the Newtonian approximation, the factor $-\eta/\varepsilon^2$ is cancelled on both sides of equations (3.18)

which, with the aid of (2.15), become

$$\begin{aligned}
(\varrho + \varepsilon^2 p)\,(u^4)^2 - \varepsilon^2 p &= -\tfrac{1}{2}\sum_{l,m,n} (\gamma_{ll,mm} + \gamma_{ll,nn}),\\
(\varrho + \varepsilon^2 p)\, u^l u^4 &= \frac{1}{2}\Big[(\gamma_{mm} + \gamma_{nn})_{,4l} + \\
&\quad + \frac{1}{\varepsilon}\,(\gamma_{4l,mm} + \gamma_{4l,nn} - \gamma_{4m,lm} - \gamma_{4n,ln})\Big],\\
(\varrho + \varepsilon^2 p)\, u^l u^m &= -\frac{1}{2\varepsilon^2}\left[(\gamma_{44} + \gamma_{nn})_{,lm} + \varepsilon\,(\gamma_{4l,m4} + \gamma_{4m,l4})\right],\\
(\varrho + \varepsilon^2 p)\,(u^l)^2 + p &= -\frac{1}{2}\Big[(\gamma_{mm} + \gamma_{nn})_{,44} - \frac{2}{\varepsilon}\,(\gamma_{4m,4m} + \gamma_{4n,4n}) - \\
&\quad - \frac{1}{\varepsilon^2}\,\big((\gamma_{44} + \gamma_{nn})_{,mm} + (\gamma_{44} + \gamma_{mm})_{,nn}\big)\Big].
\end{aligned} \tag{3.19}$$

The left hand sides of (3.19) remain finite as ε^2 tends to zero, but without introducing additional assumptions about the form of the $\gamma_{\mu\nu}$, the right hand sides do not. However, the form of the $\gamma_{\mu\nu}$ given by (3.09) is such that the right hand sides of (3.19) do remain finite as ε^2 becomes zero. For, substituting (3.09) into (3.19) gives

$$\begin{aligned}
(\varrho + \varepsilon^2 p)\,(u^4)^2 - \varepsilon^2 p &= -\nabla^2\psi - \varepsilon^2 \sum_{l,m,n} (\psi_{l,mm} + \psi_{l,nn}),\\
(\varrho + \varepsilon^2 p)\, u^4 u^l &= \psi_{,4l} + \varepsilon^2(\psi_m + \psi_n)_{,4l} + \tfrac{1}{2}\Big(\nabla^2\zeta_l - \sum_{i=1}^{3}\zeta_{i,il}\Big),\\
(\varrho + \varepsilon^2 p)\, u^l u^m &= (\psi_4 - \psi_n)_{,lm} - \tfrac{1}{2}\,(\zeta_{l,m4} + \zeta_{m,l4}),\\
(\varrho + \varepsilon^2 p)\,(u^l)^2 + p &= -\psi_{,44} - \varepsilon^2(\psi_m + \psi_n)_{,44} + \\
&\quad + (\zeta_{m,4m} + \zeta_{n,4n}) + (\psi_n - \psi_4)_{,mm} + (\psi_m - \psi_4)_{,nn}.
\end{aligned} \tag{3.20}$$

It is now possible to set ε^2 equal to zero and obtain the following Newtonian equations

$$\begin{aligned}
\varrho &= -\nabla^2\psi,\\
\varrho\, U_l &= \psi_{,4l} + \tfrac{1}{2}\Big(\nabla^2\zeta_l - \sum_{i=1}^{3}\zeta_{i,il}\Big),\\
\varrho\, U_l U_m &= (\psi_4 - \psi_n)_{,lm} - \tfrac{1}{2}\,(\zeta_{l,m4} + \zeta_{m,l4}),\\
\varrho\, U_l^2 + p &= -\psi_{,44} + \zeta_{m,4m} + \zeta_{n,4n} + (\psi_n - \psi_4)_{,mm} + (\psi_m - \psi_4)_{,nn}.
\end{aligned} \tag{3.21}$$

The restriction (3.08) has been removed, for in the present case,

$$\Omega_l = \tfrac{1}{2}\,\nabla^2(\zeta_{m,n} - \zeta_{n,m}). \tag{3.22}$$

Thus, the non-orthogonal metric

$$\begin{aligned}
d\sigma^2 &= \left[1 - \eta\,(\psi + 2\varepsilon^2\psi_4)\right](d\xi^4)^2 - \\
&\quad - 2\varepsilon^2\eta\sum_{i=1}^{3}\zeta_i\, d\xi^i d\xi^4 - \varepsilon^2\sum_{i=1}^{3}\left[1 + \eta\,(\psi + 2\varepsilon^2\psi_i)\right](d\xi^i)^2,
\end{aligned} \tag{3.23}$$

has led to a solution without the restriction (3.08) which was inherent in the orthogonal metric (3.04).

In general (McVITTIE 1956e)

$$T^{\mu\nu}_{;\nu} = 0, \tag{3.24}$$

where the semi-colon denotes convariant differentiation defined by

$$T^{\mu\nu}_{;\lambda} = T^{\mu\nu}_{,\lambda} + \left\{{\mu \atop \sigma\lambda}\right\} T^{\nu\sigma} + \left\{{\nu \atop \sigma\lambda}\right\} T^{\mu\sigma}. \tag{3.25}$$

Since terms of order η^2 have been neglected, and since, by (3.13) and (3.15), both the energy tensor and the Christoffel symbols are of order η, equations (3.24) and (3.25) imply, to this order approximation,

$$T^{\mu\nu}_{,\nu} = 0. \tag{3.26}$$

The equation of continuity and the equations of motion of a fluid under the force of its pressure gradient alone are contained in the Newtonian approximation to equations (3.26) in the following form

$$\begin{aligned} &\varrho_{,4} + \sum_{j=1}^{3} (\varrho U_j)_{,j} = 0, \\ &(\varrho U_l)_{,4} + \sum_{j=1}^{3} (\varrho U_l U_j + \delta_{lj} p)_{,j} = 0. \end{aligned} \tag{3.27}$$

Thus the expressions for Newtonian density, pressure, and velocity given by (3.21) satisfy the equations of motion and continuity (3.27) for any ψ, ψ_μ, and ζ_l, a fact which may be verified by direct substitution. However, since there are ten equations in the set (3.21) to determine only five quantities, there must be five extra equations, or consistency relations, which are relations which the ψ, ψ_μ, and ζ_l must satisfy in order that the quantities ϱ, p, and U_l be consistently defined by (3.21). These consistency relations are

$$(\psi_4 - \psi_n)_{,lm} - \frac{1}{2}(\zeta_{l,m4} + \zeta_{m,l4}) = -\frac{(\psi_{,4l} + \varphi_l)(\psi_{,4m} + \varphi_m)}{\nabla^2 \psi}, \tag{3.28}$$

$$\begin{aligned} &\zeta_{1,41} + \zeta_{2,42} + (\psi_1 - \psi_4)_{,22} + (\psi_2 - \psi_4)_{,11} + \frac{(\psi_{,43} + \varphi_3)^2}{\nabla^2 \psi} \\ &\quad = \zeta_{2,42} + \zeta_{3,43} + (\psi_2 - \psi_4)_{,33} + (\psi_3 - \psi_4)_{,22} + \frac{(\psi_{,41} + \varphi_1)^2}{\nabla^2 \psi} \\ &\quad = \zeta_{3,43} + \zeta_{1,41} + (\psi_3 - \psi_4)_{,11} + (\psi_1 - \psi_4)_{,33} + \frac{(\psi_{,42} + \varphi_2)^2}{\nabla^2 \psi}, \end{aligned} \tag{3.29}$$

where

$$\varphi_l = \tfrac{1}{2}\nabla^2 \zeta_l - \tfrac{1}{2}\sum_{i=1}^{3} \zeta_{i,il}. \tag{3.30}$$

Equations (3.28) were obtained from the first seven of equations (3.21) and may be used to express the quantities $(\psi_4 - \psi_n)_{,lm}$ in terms of ψ and ζ_l. Thus, the density, pressure, and velocity may be expressed in terms of ψ and ζ_l alone, as follows

$$\begin{aligned} \varrho &= -\nabla^2 \psi, \\ U_l &= -\frac{1}{\nabla^2 \psi}(\psi_{,4l} + \varphi_l), \\ p &= -\psi_{,44} + X, \end{aligned} \tag{3.31}$$

where X is defined by

$$X_{,l} = \sum_{j=1}^{3} \left\{ \frac{(\psi_{,4l} + \varphi_l)(\psi_{,4j} + \varphi_j)}{\nabla^2 \psi} \right\}_{,j} - \varphi_{l,4}. \tag{3.32}$$

Equations (3.29), which were obtained by setting the three expressions for $p + \psi_{,44}$ in equations (3.21) equal to each other, are the same as the requirement that equation (3.32), which defines X, should be integrable.

The hydrodynamic motions obtained above from the non-orthogonal metric, (3.23), are more general than those obtained by McVITTIE (1956b) from an orthogonal metric, but the treatment is the same. The orthogonal case can be obtained from equations (3.31) and (3.32) by setting $\varphi_l = 0$.

4. Infinitesimal coordinate transformations

In this section, the loss of generality, if any, due to using a spatially orthogonal metric will be considered, as will the relationship between orthogonality and the restriction (3.08) which makes the momentum-vorticity vanish. These questions will be discussed in terms of infinitesimal coordinate transformations.

The infinitesimal coordinate transformations from $(\hat{\xi}^4, \hat{\xi}^1, \hat{\xi}^2, \hat{\xi}^3)$ to $(\xi^4, \xi^1, \xi^2, \xi^3)$ which will be considered are defined by

$$\xi^\mu = \hat{\xi}^\mu + \eta\, f^\mu(\hat{\xi}^4, \hat{\xi}^1, \hat{\xi}^2, \hat{\xi}^3). \tag{4.01}$$

If F is any function of the coordinates, then the difference between F as a function of the $\hat{\xi}^\mu$ and F as a function of the ξ^μ is at most of order η. Thus, since the metrical tensors here considered have constant zero order terms, the difference between them as functions of the $\hat{\xi}^\mu$ and as functions of the ξ^μ is of order η^2. Terms of order η^2 are being neglected, so the distinction is not made. In the remainder of this section, then, whenever the variables of a function are not explicitly stated, they may be taken to be either set of coordinates related to each other by a transformation of the form (4.01). Following the work of CURTIS (1950), the metric may be written

$$d\sigma^2 = \hat{g}_{\mu\nu}\, d\hat{\xi}^\mu\, d\hat{\xi}^\nu, \tag{4.02}$$

$$d\sigma^2 = g_{\mu\nu}\, d\xi^\mu\, d\xi^\nu. \tag{4.03}$$

The differences between the elements of the metrical tensors $g_{\mu\nu}$ and $\hat{g}_{\mu\nu}$ are of order η, so that

$$g_{\mu\nu} = \hat{g}_{\mu\nu} + \eta\, \Delta_{\mu\nu}. \tag{4.04}$$

Equation (4.03) may be written

$$\begin{aligned} d\sigma^2 &= (\hat{g}_{\mu\nu} + \eta\, \Delta_{\mu\nu})\,(d\hat{\xi}^\mu + \eta\, f^\mu_{,\sigma}\, d\hat{\xi}^\sigma)\,(d\hat{\xi}^\nu + \eta\, f^\nu_{,\sigma}\, d\hat{\xi}^\sigma) \\ &= [\hat{g}_{\mu\nu} + \eta\,(\Delta_{\mu\nu} + \hat{g}_{\mu\sigma} f^\sigma_{,\nu} + \hat{g}_{\nu\sigma} f^\sigma_{,\mu})]\, d\hat{\xi}^\mu\, d\hat{\xi}^\nu. \end{aligned} \tag{4.05}$$

Comparing (4.05) with (4.02) gives

$$\Delta_{\mu\nu} = -\hat{g}_{\mu\sigma} f^\sigma_{,\nu} - \hat{g}_{\nu\sigma} f^\sigma_{,\mu}, \tag{4.06}$$

or

$$g_{\mu\nu} = \hat{g}_{\mu\nu} - \eta\,(\hat{g}_{\mu\sigma} f^\sigma_{,\nu} + \hat{g}_{\nu\sigma} f^\sigma_{,\mu}). \tag{4.07}$$

Using a transformation of the form (4.01) with the corresponding change in metrical tensor given by (4.07), it will be shown firstly how the metric

$$d\sigma^2 = \hat{g}_{\mu\nu}\, d\hat{\xi}^\mu\, d\hat{\xi}^\nu, \tag{4.08}$$

where

$$\begin{aligned} \hat{g}_{44} &= 1 + \eta\, \sigma_{44}, \\ \hat{g}_{4l} &= -\varepsilon\, \eta\, \sigma_{4l}, \\ \hat{g}_{ll} &= -\varepsilon^2 (1 + \eta\, \sigma_{ll}), \\ \hat{g}_{lm} &= -\varepsilon^2\, \eta\, \sigma_{lm}, \end{aligned} \tag{4.09}$$

may be transformed into a spatially orthogonal one, and secondly, how, under certain conditions, it may be transformed into an orthogonal metric. In the first case, since g_{lm} is to be zero, the corresponding equations of the set (4.07) are

$$0 = -\varepsilon^2 \eta\, \sigma_{lm} - \eta\,(-\varepsilon^2 f^l_{,m} - \varepsilon^2 f^m_{,l}), \tag{4.10}$$

or

$$\sigma_{lm} = f^l_{,m} + f^m_{,l}. \tag{4.11}$$

Let $\omega_{lm} = \omega_{ml}$ be any function such that

$$\sigma_{lm} = \omega_{lm,\,lm}; \tag{4.12}$$

then, a solution of (4.11) is

$$f^l = \tfrac{1}{2}\,(\omega_{lm} + \omega_{ln} - \omega_{mn})_{,l}. \tag{4.13}$$

If f^4 is chosen so that

$$f^4 = 0, \tag{4.14}$$

then the transformation given by (4.13) and (4.14) yields the following spatially orthogonal metrical tensor

$$\begin{aligned}
g_{44} &= 1 + \eta\, \sigma_{44}, \\
g_{4l} &= -\varepsilon\, \eta \left[\sigma_{4l} + \frac{\varepsilon}{2}\,(\omega_{mn} - \omega_{lm} - \omega_{ln})_{,4l}\right], \\
g_{ll} &= -\varepsilon^2 \left[1 + \eta\,(\sigma_{ll} + (\omega_{mn} - \omega_{lm} - \omega_{ln})_{,ll})\right], \\
g_{lm} &= 0.
\end{aligned} \tag{4.15}$$

The metrical tensor (4.15) is of the same form as (3.10). In fact

$$\begin{aligned}
\gamma_{44} &= \sigma_{44}, \\
\gamma_{4l} &= \sigma_{4l} + \frac{\varepsilon}{2}\,(\omega_{mn} - \omega_{lm} - \omega_{ln})_{,\,4l}, \\
\gamma_{ll} &= \sigma_{ll} + (\omega_{mn} - \omega_{lm} - \omega_{ln})_{,ll}.
\end{aligned} \tag{4.16}$$

Thus, it has been shown that the spatially orthogonal metrical tensor (3.10) may be obtained from the more general metrical tensor (4.09) by an infinitesimal coordinate transformation. This transformation leaves the approximate form (to order η) of Einstein's equations (3.18) unchanged.

In the second case, that of transforming (4.09) into an orthogonal metrical tensor, g_{4l} as well as g_{lm} is to be zero. Now, in addition to equations (4.10), the following equations must hold,

$$0 = -\varepsilon\,\eta\,\sigma_{4l} - \eta\,(f^4_{,l} - \varepsilon^2 f^l_{,4}), \tag{4.17}$$

or

$$f^4_{,l} = \varepsilon^2 f^l_{,4} - \varepsilon\,\sigma_{4l}. \tag{4.18}$$

Differentiating (4.18) with respect to $\hat{\xi}^m$ gives

$$f^4_{,lm} = \varepsilon^2 f^l_{,m4} - \varepsilon\,\sigma_{4l,\,m}, \tag{4.19}$$

and reversing indices gives

$$f^4_{,ml} = \varepsilon^2 f^m_{,l4} - \varepsilon\,\sigma_{4m,\,l}. \tag{4.20}$$

The sum of (4.19) and (4.20) is

$$2 f^4_{,lm} = \varepsilon^2 (f^m_{,l} + f^l_{,m})_{,4} - \varepsilon\,(\sigma_{4l,\,m} + \sigma_{4m,\,l}) = \varepsilon^2 \sigma_{lm,\,4} - \varepsilon\,(\sigma_{4l,\,m} + \sigma_{4m,\,l}). \tag{4.21}$$

If the functions $\sigma_{\mu\nu}$ are such that the three equations (4.21) define consistently a function f^4, then functions f^l can be found from (4.18) or

$$\varepsilon^2 f^l_{,4} = \varepsilon \sigma_{4l} + f^4_{,l}. \tag{4.22}$$

In order to simplify the notation, it will be assumed that the transformation to a spatially orthogonal metrical tensor (which can be made without loss of generality) has already been made. That is, $\sigma_{lm}=0$. It will next be shown that a necessary and sufficient condition on the functions σ_{4l} to insure the consistency of the equations (4.21) is

$$\sigma_{4l,mn} = \sigma_{4m,ln}. \tag{4.23}$$

The necessity of the condition follows from

$$f^4_{,lnm} = f^4_{,mnl}, \tag{4.24}$$

since, from (4.21) with $\sigma_{lm}=0$, this is equivalent to

$$\sigma_{4l,nm} + \sigma_{4n,lm} = \sigma_{4m,nl} + \sigma_{4n,ml}. \tag{4.25}$$

If, on the other hand, the condition (4.23) holds, then equations (4.25) are satisfied. Let F^l be any three functions such that

$$F^l_{,mn} = -\tfrac{1}{2}\varepsilon(\sigma_{4m,n} + \sigma_{4n,m}). \tag{4.26}$$

Equations (4.25) imply

$$F^1_{,123} = F^2_{,123} = F^3_{,123}, \tag{4.27}$$

or

$$\begin{aligned} F^1 &= F^2 + A_1(\hat{\xi}^4, \hat{\xi}^2, \hat{\xi}^3) + B_1(\hat{\xi}^4, \hat{\xi}^1, \hat{\xi}^3) + C_1(\hat{\xi}^4, \hat{\xi}^1, \hat{\xi}^2), \\ F^1 &= F^3 + A_2(\hat{\xi}^4, \hat{\xi}^2, \hat{\xi}^3) + B_2(\hat{\xi}^4, \hat{\xi}^1, \hat{\xi}^3) + C_2(\hat{\xi}^4, \hat{\xi}^1, \hat{\xi}^2). \end{aligned} \tag{4.28}$$

Let three functions G^l be defined as follows

$$\begin{aligned} G^1 &= F^1 - B_1 - C_2, \\ G^2 &= F^2 + A_1 + C_1 - C_2, \\ G^3 &= F^3 + A_2 + B_2 - B_1, \end{aligned} \tag{4.29}$$

then it is easily verified that

$$G^l_{,mn} = F^l_{,mn} = -\tfrac{1}{2}\varepsilon(\sigma_{4m,n} + \sigma_{4n,m}), \tag{4.30}$$

and

$$G^1 = G^2 = G^3. \tag{4.31}$$

If f^4 is taken to be the function which is equal to each of the three functions G^l, then the sufficiency of the condition is established.

In Section 3 it was shown that the orthogonal metric (3.04) led to the vanishing of the momentum-vorticity. It was also shown that the non-orthogonal metric (3.23) removed that restriction on the motion. However, the condition (4.23) for the existence of an infinitesimal coordinate transformation which transforms the metrical tensor (3.10) into an orthogonal one is neither necessary nor sufficient for the vanishing of the momentum-vorticity. To show this two examples will be used. The first will be a case where the metrical tensor can be transformed into an orthogonal one and the momentum-vorticity is non-zero,

and the second example will be one where the metrical tensor cannot be so transformed but the momentum-vorticity does vanish.

The condition (4.23), in the case of the metrical tensor (3.10) is

$$\zeta_{l,mn} = \zeta_{m,ln}. \tag{4.32}$$

For the sake of an example, consider the case where only one of the ζ_l, say ζ_1, is different from zero. The condition (4.32) becomes

$$\zeta_{1,23} = 0. \tag{4.33}$$

Following the procedure indicated above, a function $\omega_{14} = \omega_{41}$ is chosen such that both

$$\omega_{14,14} = \zeta_1, \tag{4.34}$$

and

$$\omega_{14,23} = 0. \tag{4.35}$$

Then, the transformation defined by

$$\begin{aligned} f^4 &= -\frac{\varepsilon^2}{2}\omega_{14,4}, \\ f^1 &= \tfrac{1}{2}\omega_{14,1}, \\ f^2 &= -\tfrac{1}{2}\omega_{14,2}, \\ f^3 &= -\tfrac{1}{2}\omega_{14,3}, \end{aligned} \tag{4.36}$$

transforms the metric (3.10), with $\zeta_2 = \zeta_3 = 0$, into the orthogonal metric

$$d\sigma^2 = \sum_{\mu=1}^{4} g^*_{\mu\mu}(d\xi^{*\mu})^2, \tag{4.37}$$

where

$$\begin{aligned} g^*_{44} &= 1 - \eta(\psi + 2\varepsilon^2\psi_4 - \varepsilon^2\omega_{14,44}), \\ g^*_{11} &= -\varepsilon^2[1 + \eta(\psi + 2\varepsilon^2\psi_1 - \omega_{14,11})], \\ g^*_{22} &= -\varepsilon^2[1 + \eta(\psi + 2\varepsilon^2\psi_2 + \omega_{14,22})], \\ g^*_{33} &= -\varepsilon^2[1 + \eta(\psi + 2\varepsilon^2\psi_3 + \omega_{14,33})]. \end{aligned} \tag{4.38}$$

The first order terms of Einstein's equations have not been affected by the above transformation, nor has the Newtonian approximation. Even though the metrical tensor is orthogonal, there are two nonvanishing terms of the momentum-vorticity, namely

$$\begin{aligned} (\varrho U_1)_{,2} - (\varrho U_2)_{,1} &= \tfrac{1}{2}\nabla^2\zeta_{1,2} = \tfrac{1}{2}\nabla^2\omega_{14,142}, \\ (\varrho U_1)_{,3} - (\varrho U_3)_{,1} &= \tfrac{1}{2}\nabla^2\zeta_{1,3} = \tfrac{1}{2}\nabla^2\omega_{14,143}. \end{aligned} \tag{4.39}$$

For the second example, take

$$\begin{aligned} \zeta_1 &= \xi^2 \cdot \xi^3, \\ \zeta_2 &= \zeta_3 = 0. \end{aligned} \tag{4.40}$$

In this case, since $\zeta_{1,23} \neq 0$, the transformation to an orthogonal metric cannot be made, but the momentum-vorticity clearly vanishes. Thus there is no intrinsic connection between the orthogonality of the metric and the vanishing of the momentum-vorticity.

5. On an approximation due to Einstein & Infeld

EINSTEIN & INFELD (1949) have developed a method of successive approximations in terms of what they call an arbitrary small parameter λ. The purpose of the present section is to investigate the significance of λ in terms of the approximations used in previous sections of this work.

The metric (2.05) may be written in terms of general relativity units, for comparison with the work of EINSTEIN & INFELD, as follows

$$[d(c\,t\,\sigma)]^2 = (1+\eta\,\gamma_{44})\,[d(c\,t\,\xi^4)]^2 - 2\sum_{i=1}^{3}\eta\,\gamma_{4i}\,d(l\,\xi^i)\,d(c\,t\,\xi^4) - \sum_{i=1}^{3}(1+\eta\,\gamma_{ii})\,[d(l\,\xi^i)]^2. \tag{5.01}$$

EINSTEIN & INFELD used the metric (in general relativity units)

$$d\,s^2 = (1+h_{00})\,(d\,x^0)^2 + 2\sum_{i=1}^{3}h_{0i}\,d\,x^0\,d\,x^i - \sum_{i,j}(\delta_{ij}-h_{ij})\,d\,x^i\,d\,x^j. \tag{5.02}$$

If the two metrics (5.01) and (5.02) are to describe the same space, it must be the case that

$$\begin{aligned} c\,t\,\sigma &= s, \\ c\,t\,\xi^4 &= x^0, \\ l\,\xi^i &= x^i, \\ \eta\,\gamma_{44} &= h_{00}, \\ -\eta\,\gamma_{4i} &= h_{0i}, \\ -\eta\,\delta_{ij}\gamma_{ii} &= h_{ij}. \end{aligned} \tag{5.03}$$

The parameter λ is introduced as a scale factor by the introduction of an auxiliary time τ, which, in terms of the time coordinate x^0 of (5.02), is

$$\tau = x^0\,\lambda. \tag{5.04}$$

All computations are carried out in terms of the coordinates (τ, x^1, x^2, x^3) rather than with (x^0, x^1, x^2, x^3). The reason EINSTEIN & INFELD give for doing so is that derivatives of functions with respect to τ can then be treated on the same footing as the space derivatives. In Section 3 of the present work, all calculations were carried out in terms of the coordinates $(\xi^4, \xi^1, \xi^2, \xi^3)$ and all derivatives were assumed to be of the same order of magnitude. Thus the systems (τ, x^1, x^2, x^3) and $(\xi^4, \xi^1, \xi^2, \xi^3)$ can be identified because all derivatives are treated on the same footing in both systems. Now since $l\,\xi^i = x^i$, it must also be the case that

$$l\,\xi^4 = \tau. \tag{5.05}$$

Combining (5.04) with (5.05) and $c\,t\,\xi^4 = x^0$ gives

$$\lambda = \frac{\tau}{x^0} = \frac{l\,\xi^4}{c\,t\,\xi^4} = \frac{l/t}{c} = \frac{v}{c} = \varepsilon. \tag{5.06}$$

Thus the coordinate system used in Section 3 forces the identification of λ with ε.

EINSTEIN & INFELD expand their $h_{\mu\nu}$ in powers of λ as follows

$$\begin{aligned} h_{00} &= \lambda^2\left(\tfrac{1}{2}\underset{2}{\gamma}_{00}\right) + \lambda^4\left(\tfrac{1}{2}\underset{4}{\gamma}_{00} + \tfrac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss}\right) + \cdots, \\ h_{0i} &= \lambda^3\underset{3}{\gamma}_{0i} + \lambda^5\underset{5}{\gamma}_{0i} + \cdots, \\ h_{ij} &= \lambda^2\left(\tfrac{1}{2}\delta_{ij}\underset{2}{\gamma}_{00}\right) + \lambda^4\left(\underset{4}{\gamma}_{ij} - \tfrac{1}{2}\delta_{ij}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss} + \tfrac{1}{2}\delta_{ij}\underset{4}{\gamma}_{00}\right) + \cdots, \end{aligned} \tag{5.07}$$

where the above $\underset{\varkappa}{\gamma}_{\mu\nu}$ are not to be confused with the $\gamma_{\mu\nu}$ used heretofore in this work. Combining (5.03) with (5.07) gives

$$\begin{aligned} \eta\gamma_{44} &= \lambda^2\left(\tfrac{1}{2}\underset{2}{\gamma}_{00}\right) + \lambda^4\left(\tfrac{1}{2}\underset{4}{\gamma}_{00} + \tfrac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss}\right) + \cdots, \\ -\eta\gamma_{4i} &= \lambda^3\underset{3}{\gamma}_{0i} + \lambda^5\underset{5}{\gamma}_{0i} + \cdots, \\ -\eta\gamma_{ii} &= \lambda^2\left(\tfrac{1}{2}\underset{2}{\gamma}_{00}\right) + \lambda^4\left(\underset{4}{\gamma}_{ii} - \tfrac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss} + \tfrac{1}{2}\underset{4}{\gamma}_{00}\right) + \cdots. \end{aligned} \tag{5.08}$$

In Section 3, the $\gamma_{\mu\nu}$ were expanded in powers of ε in the following way:

$$\begin{aligned} \gamma_{44} &= -(\psi + 2\varepsilon^2\psi_4), \\ \gamma_{4i} &= \varepsilon\zeta_i, \\ \gamma_{ii} &= \psi + 2\varepsilon^2\psi_i. \end{aligned} \tag{5.09}$$

Substituting from (5.09) into (5.08) gives

$$\begin{aligned} -\eta\psi - 2\eta\,\varepsilon^2\psi_4 &= \lambda^2\left(\tfrac{1}{2}\underset{2}{\gamma}_{00}\right) + \lambda^4\left(\tfrac{1}{2}\underset{4}{\gamma}_{00} + \tfrac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss}\right) + \cdots, \\ -\eta\,\varepsilon\zeta_i &= \lambda^3\underset{3}{\gamma}_{0i} + \cdots, \\ -\eta\psi - 2\eta\,\varepsilon^2\psi_i &= \lambda^2\left(\tfrac{1}{2}\underset{2}{\gamma}_{00}\right) + \lambda^4\left(\underset{4}{\gamma}_{ii} + \tfrac{1}{2}\underset{4}{\gamma}_{00} - \tfrac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss}\right) + \cdots. \end{aligned} \tag{5.10}$$

Equations (5.10) are consistent with the identification $\lambda = \varepsilon$ together with

$$\begin{aligned} -\frac{\eta}{\varepsilon^2}\psi &= \frac{1}{2}\underset{2}{\gamma}_{00}, \\ -2\frac{\eta}{\varepsilon^2}\psi_4 &= \frac{1}{2}\underset{4}{\gamma}_{00} + \frac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss}, \\ -\frac{\eta}{\varepsilon^2}\zeta_i &= \underset{3}{\gamma}_{0i}, \\ -2\frac{\eta}{\varepsilon^2}\psi_i &= \underset{4}{\gamma}_{ii} + \frac{1}{2}\underset{4}{\gamma}_{00} - \frac{1}{2}\sum_{s=1}^{3}\underset{4}{\gamma}_{ss}. \end{aligned} \tag{5.11}$$

Thus the method used in Section 3 would appear to be equivalent to working to the order λ^4 by the method of EINSTEIN & INFELD.

The identification $\lambda = \varepsilon$ was dictated by the choice of coordinate system. The parameter λ is said by EINSTEIN & INFELD to be arbitrary, which would suggest that other identifications should be possible. At first glance, it would seem

possible to identify λ with $\eta^{\frac{1}{2}}$, but that cannot be done consistently without leading to the equality of ε^2 and η, which, in turn, leads back to the original identification (5.06). Thus, the parameter λ is only as arbitrary as ε is, which means that it can be varied only so far as the ratio v of the scale factors l and t can be.

6. Notation

σ dimensionless length of arc.
$g_{\mu\nu}$ dimensionless metrical tensor.
ξ^μ dimensionless coordinates.
ε small dimensionless parameter inversely proportional to velocity of light.
η small dimensionless parameter proportional to constant of gravitation.
$R_{\mu\nu}$ dimensionless Ricci tensor.
$T^{\mu\nu}$ dimensionless energy tensor.
$\gamma_{\mu\nu}, \psi, \psi_\mu, \zeta_l$ functions of coordinates used in expansion of metrical tensor in terms of small parameters.
ϱ dimensionless density.
p dimensionless pressure.
U_l dimensionless Newtonian velocity.
Ω_l momentum-vorticity.

Greek indices range over the values 4, 1, 2, 3 and Latin indices over the values 1, 2, 3 only. The indices l, m, n always denote a cyclic permutation of 1, 2, 3. A comma denotes partial differentiation with respect to the coordinate(s) indicated by the subscript(s) after the comma.

Other symbols are used in one section only and are explained where they are used.

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The Determination of the Inverse Matrix for a Basic Reference Equation for the Theory of Hydrodynamic Stability

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1. Introduction

In the development of an asymptotic theory for the Orr-Sommerfeld equation of hydrodynamic stability, the equation [*1*]

$$u^{iv} + \lambda^2 (z u'' + \alpha u' + \beta u) = 0 \tag{1.1}$$

plays the role of a reference equation. This equation is equivalent to the first order system

$$\frac{d}{dz} U = A_0(z, \lambda)\, U \tag{1.2}$$

where $A_0(z, \lambda)$ is the matrix

$$A_0(z, \lambda) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ -\beta & -\alpha & -\lambda z & 0 \end{bmatrix}. \tag{1.3}$$

If u_i, $i=1, 2, 3, 4$, are the components of a column vector of a matrix solution $U(z, \lambda)$ of (1.2), these components are related to a solution u of (1.1) by the equations

$$u_1 = u, \qquad u_2 = u', \qquad u_3 = \lambda^{-1} u'', \qquad u_4 = \lambda^{-2} u'''. \tag{1.4}$$

For every fundamental matrix solution U of (1.2), the matrix $(U^{-1})^T$ is a fundamental matrix solution of the adjoint system

$$\frac{d}{dz} V = - A_0^T(z, \lambda)\, V. \tag{1.5}$$

The adjoint of (1.1) is

$$v^{iv} + \lambda^2 [z v'' + (2-\alpha) v' + \beta v] = 0. \tag{1.6}$$

This has the same form as (1.1), the only difference being that α is replaced by $(2-\alpha)$. Hence the asymptotic forms of its solutions, for large values of $|\lambda|$, are known from [*1*]. The components v_i, $i=1, 2, 3, 4$, of a vector solution of

(1.5) are related to a solution v of (1.6) by the equations

$$\begin{aligned} v_1 &= (\alpha-1)\,v - z\,v' - \lambda^{-2} v''', \\ v_2 &= z\,v + \lambda^{-2} v'', \\ v_3 &= -\lambda^{-1} v', \\ v_4 &= v. \end{aligned} \tag{1.7}$$

In particular, the elements in the last column of the matrix U^{-1}, corresponding to a fundamental matrix U for (1.2), are solutions of (1.6).

The hydrodynamical equation [1] is of the form

$$\varphi^{iv} + \lambda^2 \sum_{n=0}^{2} P_n(y)\,\varphi^{(2-n)} + \sum_{n=0}^{2} Q_n(y,\lambda)\,\varphi^{(2-n)} = 0. \tag{1.8}$$

From any given solution of the reference equation (1.1) may be constructed a formal expression which can be shown to represent a true solution of (1.8) asymptotically in a certain region of the complex y-plane. The details of this analysis will be presented in another paper. However, in order to carry out the proof of the above statement, it turns out to be necessary to know the asymptotic behavior, for large $|\lambda|$, of the inverse matrices corresponding to various fundamental matrix solutions of (1.2).

In this article we shall derive formulae which will enable us to express the elements of the inverse matrix U^{-1}, corresponding to any given fundamental matrix U of (1.2), in terms of specific solutions of (1.6). The asymptotic behavior of these elements will then be known from [1]. Furthermore, this explicit knowledge of the elements of U^{-1} in terms of the solutions of (1.6) makes it convenient for us to write the asymptotic expansions of the elements in different forms for different sectors of the complex z-plane. This can be done by using the relations (2.6) of [1], with α replaced by $(2-\alpha)$. This turns out to be necessary in the analysis of (1.8).

The principal results of this article are the formulae given in Section 4. These express the third order Wronskians of the solutions of (1.1) in terms of the solutions of (1.6). The use of these formulae may be illustrated by the following example. Let U be the fundamental matrix

$$U = \begin{bmatrix} B_0(z,\alpha) & B_1(z,\alpha) & B_2(z,\alpha) & B_3(z,\alpha) \\ B_0' & B_1' & B_2' & B_3' \\ \lambda^{-1} B_0'' & \lambda^{-1} B_1'' & \lambda^{-1} B_2'' & \lambda^{-1} B_3'' \\ \lambda^{-2} B_0''' & \lambda^{-2} B_1''' & \lambda^{-2} B_2''' & \lambda^{-2} B_3''' \end{bmatrix}. \tag{1.9}$$

Here the functions $B_i(z,\alpha)$, $i=0, 1, 2, 3$, are specific solutions of (1.1) defined in [1]. Suppose it is desired to know the asymptotic behavior of the elements of the inverse matrix U^{-1}. By direct calculation we find that

$$U^{-1} = \frac{\lambda^2}{W(B_0, B_1, B_2, B_3)} \begin{bmatrix} - & - & - & -W(B_1, B_2, B_3) \\ - & - & - & W(B_0, B_2, B_3) \\ - & - & - & -W(B_0, B_1, B_3) \\ - & - & - & W(B_0, B_1, B_2) \end{bmatrix}. \tag{1.10}$$

By the use of formulae (4.32)—(4.35) at the end of this article, it follows that

$$(1.11)\qquad U^{-1} = \frac{-2\pi i\, e^{-\pi i\alpha}\, \beta^{\alpha-1}\, \lambda^{2\alpha+2}}{W(B_0, B_1, B_2, B_3)} \begin{bmatrix} - & - & - & -e^{2\pi i\alpha} B_2(z, 2-\alpha) \\ - & - & - & A_1(z, 2-\alpha) \\ - & - & - & A_2(z, 2-\alpha) \\ - & - & - & e^{2\pi i\alpha} A_3(z, 2-\alpha) \end{bmatrix}.$$

The first three columns in the matrix here may be determined by means of the relations (1.7), although for the analysis of (1.8) these are not needed. The Wronskian $W(B_0, B_1, B_2, B_3)$ is independent of z, since (1.1) has no third derivative term. It may be evaluated by the use of the formulae derived in Section 2, which give the behavior of the solutions for large values of $|z|$. It is found that

$$(1.12)\qquad W(B_0, B_1, B_2, B_3) = 4\pi^2 \beta^{\alpha-1} \lambda^{2\alpha+2}.$$

The inverse of any other fundamental matrix may be found in the same manner by the use of the appropriate formulae in Section 4.

2. The asymptotic properties of the solutions for large values of $|z|$

The known behavior of the solutions of (1.1) for large values of $|\lambda|$, while sufficing to determine the asymptotic properties of the elements of the inverse matrix, are not sufficient to establish exact formulae expressing these elements in terms of the solutions of (1.6). It is true that only the asymptotic behavior, for large $|\lambda|$, of these elements is needed in the study of (1.8). However, for the sake of a mathematically complete discussion of the equation (1.1), we shall prove the exactitude of the relations listed in Section 4. Furthermore, as mentioned in the Introduction, these relations greatly facilitate the writing of the asymptotic expressions for the elements of the inverse matrix in their most useful forms. In order to prove the relations, we need the leading terms of the asymptotic forms of the solutions of (1.1) and (1.6) for the case when λ is fixed and the independent variable z is large in magnitude. These will be derived in this section. The establishment of the relations (4.1)—(4.35) will be carried out in Section 3.

(i) *The functions A_k.*

In Section 4 of [*1*], the functions A_k were expressed in terms of the variable $\xi = \frac{2}{3} i \lambda z^{\frac{3}{2}}$. They were then evaluated by the method of steepest descents, the asymptotic forms (4.21) being valid for large values of $|\xi|$. In [*1*], z was restricted to a bounded domain and λ was large in magnitude. However, in the case where $|\xi|$ becomes large with λ fixed (and different from zero) and $|z|$ large, the method of steepest descents still applies. In this case, an examination of the calculations involved shows that the error term $O(\xi^{-1})$ must be replaced by the term $O(|z|^{-\frac{3}{2}})$. Thus we have

$$(2.1)\qquad A_k(z, \alpha) = i\sqrt{\pi}\,(-1)^k\, e^{-\frac{3}{2}\pi i(\frac{1}{3}\alpha+\frac{1}{2})}\, \lambda^{\alpha-\frac{3}{2}}\, z^{\frac{1}{2}\alpha-\frac{5}{4}}\, e^{-\frac{2}{3} i\lambda z^{\frac{3}{2}}} \left[1 + O(|z|^{-\frac{3}{2}})\right]$$

with

$$(2.1\,a)\qquad \tfrac{4}{3}\pi(k-2) - \tfrac{2}{3}\arg\lambda < \arg z < 2\pi + \tfrac{4}{3}\pi(k-2) - \tfrac{2}{3}\arg\lambda,$$

for z in $S - C_k$.

(ii) *The function* B_0.

From Section 3 of [1] we have

$$B_0 = \int_{C(B_0)} t^{\alpha-2} \exp(z t - \beta t^{-1}) \exp(\tfrac{1}{3} \lambda^{-2} t^3)\, dt. \tag{2.2}$$

Let

$$t = \beta^{\frac{1}{2}} z^{-\frac{1}{2}} \tau, \qquad \zeta = 2\beta^{\frac{1}{2}} z^{\frac{1}{2}}.$$

Then

$$B_0 = \left(\frac{2\beta}{\zeta}\right)^{\alpha-1} \int_{C'(B_0)} \tau^{\alpha-2} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] \exp\left[\frac{1}{3}\lambda^{-2}\left(\frac{2\beta}{\zeta}\right)^3 \tau^3\right] d\tau. \tag{2.3}$$

The path of integration in the τ-plane emerges from the origin with direction $\arg\tau = \arg\zeta - 2\pi$, circles the origin, and tends to zero again with direction $\arg\tau = \arg\zeta$.

For the purpose of evaluating B_0 for large $|\zeta|$, we write

$$\exp\left[\frac{1}{3}\lambda^{-2}\left(\frac{2\beta}{\zeta}\right)^3 \tau^3\right] = 1 + \zeta^{-3}\tau^3 h(\tau, \zeta, \lambda), \tag{2.4}$$

where $h(\tau, \zeta, \lambda)$ is uniformly bounded for fixed $\lambda \neq 0$, $|\zeta| \geqq |\zeta_0| > 0$, and $|\tau| < |\tau_0|$. Then

$$\begin{aligned} B_0 = \left(\frac{2\beta}{\zeta}\right)^{\alpha-1} \Bigg\{ &\int_{C'(B_0)} \tau^{\alpha-2} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] d\tau + \\ &+ \zeta^{-3} \int_{C'(B_0)} \tau^{\alpha+1} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] h(\tau, \zeta, \lambda)\, d\tau \Bigg\}. \end{aligned} \tag{2.5}$$

For the first integral here we have ([2], p. 199)

$$\int_{C'(B_0)} \tau^{\alpha-2} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] d\tau = -2\pi i\, e^{-\pi i \alpha} J_{\alpha-1}(\zeta), \tag{2.6}$$

where

$$J_{\alpha-1}(\zeta) = \frac{1}{2}\left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}} \{e^{i(\zeta - \frac{1}{2}\pi\alpha + \frac{1}{4}\pi)}[1 + O(|\zeta|^{-1})] + e^{-i(\zeta - \frac{1}{2}\pi\alpha + \frac{1}{4}\pi)}[1 + O(|\zeta|^{-1})]\} \tag{2.7}$$

for $|\zeta|$ large and for $|\arg\zeta| < \pi$.

There remains the task of estimating the second integral in (2.5). For this purpose we write

$$\int_{C'(B_0)} \tau^{\alpha+1} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] h(\tau, \zeta, \lambda)\, d\tau = I_1 + I_2. \tag{2.8}$$

The integral I_j, $j = 1, 2$, corresponds to the path $C(I_j)$ shown in Figure 1. The function $\exp[\zeta f(\tau)]$, where $f(\tau) = \frac{1}{2}\left(\tau - \frac{1}{\tau}\right)$, has saddle points at $\tau = \pm i$. The path of steepest descent through $\tau = i$ leaves the origin with direction $\arg\tau = \arg\zeta - 2\pi$ and goes to infinity with direction $\arg\tau = -\pi - \arg\zeta$ for $-\frac{1}{2}\pi < \arg\zeta < \frac{3}{2}\pi$. The path of steepest descent through $\tau = -i$ leaves the origin with direction $\arg\tau = \arg\zeta$ and goes to infinity with direction $\arg\tau = -\pi - \arg\zeta$

for $-\frac{3}{2}\pi < \arg\zeta < \frac{1}{2}\pi$. Suppose for the moment we take $|\arg\zeta| < \frac{1}{2}\pi$ and consider I_1. Let $C(I_1)$ follow the path of steepest descent through $\tau = i$ to a point where $\zeta[f(\tau) - f(i)] = \zeta f(\tau) - i\zeta$ is real and negative, and then let it be joined to $C(I_2)$ by a part along which $\mathrm{Re}\{\zeta[f(\tau) - f(i)]\}$ is constant and negative. Thus if $|\arg\zeta| \leqq \pi - \delta$, the path $C(I_1)$ can be chosen so that $e^{-i\zeta} I_1$ is uniformly bounded for $|\zeta| \geqq |\zeta_0| > 0$. Similarly, with $C(I_2)$ being chosen to follow the path of steepest descent through $\tau = -i$, $e^{i\zeta} I_2$ is uniformly bounded for $|\arg\zeta| \leqq \pi - \delta$.

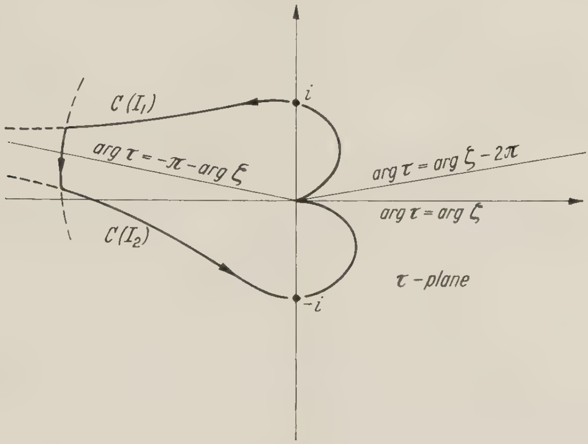

Fig. 1

There now follows from (2.5), (2.6), and (2.7) that

$$B_0 = -i\sqrt{\pi}\, e^{-\pi i\alpha} \beta^{-\frac{1}{2}} \left(\frac{2\beta}{\zeta}\right)^{\alpha-\frac{1}{2}} \{e^{i(\zeta - \frac{1}{2}\pi\alpha + \frac{1}{4}\pi)}[1 + O(|\zeta|^{-1})] + e^{-i(\zeta - \frac{1}{2}\pi\alpha + \frac{1}{4}\pi)}[1 + O(|\zeta|^{-1})]\}, \tag{2.9}$$

where $|\arg\zeta| < \pi$, or

$$B_0 = -i\sqrt{\pi}\, e^{-\pi i\alpha} \beta^{\frac{1}{2}\alpha - \frac{3}{4}} z^{\frac{1}{4} - \frac{1}{2}\alpha} \{e^{i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}} - \frac{1}{2}\pi\alpha + \frac{1}{4}\pi)}[1 + O(|z|^{-\frac{1}{2}})] + e^{-i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}} - \frac{1}{2}\pi\alpha + \frac{1}{4}\pi)}[1 + O(|z|^{-\frac{1}{2}})]\}, \tag{2.10}$$

where $|\arg\beta z| < 2\pi$.

(iii) *The function B_k.*

We shall discuss in detail the solution B_1. The behavior of B_2 and B_3 can be found in an entirely similar manner. From [*1*] we have

$$B_1 = \int_{C(B_1)} t^{\alpha-2} \exp(zt - \beta t^{-1}) \exp(\tfrac{1}{3}\lambda^{-2} t^3)\, dt. \tag{2.11}$$

As mentioned in Section 5 of [*1*], it is possible to choose the path of integration so that $\mathrm{Re}(tz) < 0$ and $\mathrm{Re}(\lambda^{-2} t^3) < 0$ for large $|t|$, provided that z lies in the sector $S - S_1$. Now let

$$\tau = t z^{\frac{1}{2}} \beta^{-\frac{1}{2}}, \quad \zeta = 2\beta^{\frac{1}{2}} z^{\frac{1}{2}}, \tag{2.12}$$

where

$$-\tfrac{2}{3}\arg\lambda < \arg z < \tfrac{4}{3}\pi - \tfrac{2}{3}\arg\lambda. \tag{2.13}$$

Then

$$B_1 = \left(\frac{2\beta}{\zeta}\right)^{\alpha-1} \int\limits_{C'(B_1)} \tau^{\alpha-2} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] \exp\left[\frac{1}{3}\lambda^{-2}\left(\frac{2\beta}{\zeta}\right)^3 \tau^3\right] d\tau \tag{2.14}$$

or

$$B_1 = \left(\frac{2\beta}{\zeta}\right)^{\alpha-1} \Bigg\{ \int\limits_{C'(B_1)} \tau^{\alpha-2} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] d\tau + \\ + \zeta^{-3} \int\limits_{C'(B_1)} \tau^{\alpha+1} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] h(\tau, \zeta, \lambda)\, d\tau \Bigg\}, \tag{2.15}$$

where $h(\tau, \zeta, \lambda)$ is bounded on the path of integration for $|\zeta| \geqq |\zeta_0| > 0$ and for fixed λ with $\lambda \neq 0$.

Here $\arg\zeta$ lies in the range

$$\tfrac{1}{2}\arg\beta - \tfrac{1}{3}\arg\lambda < \arg\zeta < \tfrac{2}{3}\pi + \tfrac{1}{2}\arg\beta - \tfrac{1}{3}\arg\lambda. \tag{2.16}$$

Also, for large $|\tau|$, we have

$$\tfrac{1}{2}\pi < \arg\tau + \arg\zeta < \tfrac{3}{2}\pi \tag{2.17}$$

while near $\tau = 0$,

$$-\tfrac{1}{2}\pi < \arg\tau - \arg\zeta < \tfrac{1}{2}\pi. \tag{2.18}$$

In [*1*] we restricted $\arg\beta$ to the range

$$\tfrac{2}{3}\arg\lambda \leqq \arg\beta < 2\pi + \tfrac{2}{3}\arg\lambda. \tag{2.19}$$

If $\arg\beta$ is allowed to vary through this range, it follows from (2.16) that

$$0 < \arg\zeta < \tfrac{5}{3}\pi. \tag{2.20}$$

In the appraisal of the second integral in (2.15), the restriction that $\arg\zeta$ lie in the range $(-\pi, 2\pi)$ is necessary, and this is the case when (2.20) holds.

For the first integral in (2.15) we have

$$\int\limits_{C'(B_1)} \tau^{\alpha-2} \exp\left[\frac{1}{2}\zeta\left(\tau - \frac{1}{\tau}\right)\right] d\tau = \pi i H^{(1)}_{1-\alpha}(\zeta), \tag{2.21}$$

where

$$H^{(1)}_{1-\alpha}(\zeta) = \left(\frac{2}{\pi\zeta}\right)^{\frac{1}{2}} e^{i(\zeta + \frac{1}{2}\pi\alpha - \frac{3}{4}\pi)} \left[1 + O(|\zeta|^{-1})\right] \tag{2.22}$$

([*2*], p. 197). This last relation is valid for $\arg\zeta$ in the range $(-\pi, 2\pi)$, and hence for $\arg\zeta$ in the range (2.16).

Next let us consider

$$e^{-i\zeta} \int\limits_{C'(B_1)} \tau^{\alpha+1} e^{\zeta f(\tau)} h(\tau, \zeta, \lambda)\, d\tau, \tag{2.23}$$

where $f(\tau) = \frac{1}{2}\left(\tau - \frac{1}{\tau}\right)$. For $\arg\zeta$ in the range (2.16), the path $C'(B_1)$ can be taken so as to go to infinity in a sector where $\operatorname{Re}[\zeta f(\tau)] < 0$, and where $h(\tau, \zeta, \lambda)$ is bounded. The function $f(\tau)$ has a saddle point at $\tau = i$. The path of steepest descent through $\tau = i$ for the function $\exp[\zeta f(\tau)]$ tends to zero with direction

$\arg \tau = \arg \zeta$ and goes to infinity with direction $\arg \tau = \pi - \arg \zeta$ for $-\frac{1}{2}\pi < \arg \zeta < \frac{2}{3}\pi$. We may therefore choose the path of integration in (2.23) as follows. Starting at the origin, we follow the path of steepest descent through $\tau = i$ to a point where $\zeta[f(\tau) - f(i)]$ is real and negative. We then take a circular arc $|\tau| =$ constant into the sector where $h(\tau, \zeta, \lambda)$ is bounded, and then a ray going to infinity in the sector where $\operatorname{Re}[\zeta f(\tau) - i\zeta]$ is less than zero and $h(\tau, \xi, \lambda)$ is bounded. On the finite part of the path which follows the path of steepest descent and the circular arc, $e^{-i\zeta} e^{\zeta f(\tau)}$ will be uniformly bounded for $|\zeta| \geqq |\zeta_0| > 0$ and $-\pi + \delta \leqq \arg \zeta \leqq 2\pi - \delta$. On the ray extending to infinity, $\zeta f(\tau) - i\zeta$ will have a negative real part and $h(\tau, \zeta, \lambda)$ will be bounded for $\arg \zeta$ in the range (2.16). Then for $\arg \zeta$ in any closed subinterval of (2.16), and for $|\zeta| \geqq |\zeta_0| > 0$, the function in (2.23) will be uniformly bounded.

It follows now from (2.15), (2.21), and (2.22) that

$$B_1 = i\sqrt{\pi}\,\beta^{-\frac{1}{2}}\left(\frac{2\beta}{\zeta}\right)^{\alpha-\frac{1}{2}} e^{i(\zeta+\frac{1}{2}\pi\alpha-\frac{3}{4}\pi)}\left[1 + O(|\zeta|^{-1})\right], \tag{2.24}$$

where

$$\tfrac{1}{2}\arg\beta - \tfrac{1}{3}\arg\lambda < \arg\zeta < \tfrac{2}{3}\pi + \tfrac{1}{2}\arg\beta - \tfrac{1}{3}\arg\lambda, \tag{2.24a}$$

or

$$B_1 = i\sqrt{\pi}\,\beta^{\frac{1}{2}\alpha-\frac{3}{4}} z^{\frac{1}{4}-\frac{1}{2}\alpha} e^{i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}}+\frac{1}{2}\pi\alpha-\frac{3}{4}\pi)}\left[1 + O(|z|^{-\frac{1}{2}})\right], \tag{2.25}$$

where

$$-\tfrac{2}{3}\arg\lambda < \arg z < \tfrac{4}{3}\pi - \tfrac{2}{3}\arg\lambda. \tag{2.25a}$$

The results for B_2 and B_3 are

$$B_2 = i\sqrt{\pi}\,\beta^{\frac{1}{2}\alpha-\frac{3}{4}} z^{\frac{1}{4}-\frac{1}{2}\alpha} e^{i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}}+\frac{1}{2}\pi\alpha-\frac{3}{4}\pi)}\left[1 + O(|z|^{-\frac{1}{2}})\right], \tag{2.26}$$

$$B_3 = i\sqrt{\pi}\,e^{-2\pi i\alpha}\beta^{\frac{1}{2}\alpha-\frac{3}{4}} z^{\frac{1}{4}-\frac{1}{2}\alpha} e^{i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}}+\frac{1}{2}\pi\alpha-\frac{3}{4}\pi)}\left[1 + O(|z|^{-\frac{1}{2}})\right] \tag{2.27}$$

where

$$-\tfrac{2}{3}\pi - \tfrac{2}{3}\pi(k-2) - \tfrac{2}{3}\arg\lambda < \arg z < \tfrac{2}{3}\pi - \tfrac{2}{3}\pi(k-2) - \tfrac{2}{3}\arg\lambda, \tag{2.28}$$

for z in $S - S_k$.

3. Derivation of the Wronskian relationships

As was shown in the introduction, the elements of the inverse of any fundamental matrix for (1.1) may be expressed in terms of specific solutions of (1.6) by the use of the formulae listed in Section 4. These formulae express the third order Wronskians of solutions of (1.1) in terms of the solutions of (1.6). In order to establish the validity of these formulae, we first show that the relations (4.1), (4.7), (4.8), and (4.12) hold. Since the four solutions involved in these relations constitute a fundamental set for (1.6), the other formulae may be obtained by expressing the other Wronskians in terms of these four by means of the relations

$$\begin{aligned} \overline{B}_2 + A_1 &= B_3, \\ B_3 + A_2 &= B_1 + B_0 \qquad (\overline{B}_2 = e^{-2\pi i\alpha} B_2), \\ B_1 + A_3 &= B_2, \end{aligned} \tag{3.1}$$

which were derived in [*1*].

Consider first the relation (4.7). Taking first z in $S-S_1$ with $-\frac{2}{3}\arg\lambda<\arg z<\frac{4}{3}\pi-\frac{2}{3}\arg\lambda$, and using the results of Section 2, we have

$$
\begin{aligned}
(3.2)\qquad W(A_2,A_3,B_1)(z,\alpha) &= \pi\,\lambda^{2\alpha-3}\,z^{\alpha-\frac{5}{2}}\begin{vmatrix} 1 & 1 & B_1(z,\alpha)\\ -i\lambda z^{\frac{1}{2}} & i\lambda z^{\frac{1}{2}} & B_1'(z,\alpha)\\ -\lambda^2 z & -\lambda^2 z & B_1''(z,\alpha)\end{vmatrix}\left[1+O\left(|z|^{-\frac{1}{2}}\right)\right]\\
&= -2\pi^{\frac{3}{2}}\beta^{\frac{1}{2}\alpha-\frac{3}{4}}\lambda^{2\alpha}z^{\frac{1}{2}\alpha-\frac{3}{4}}e^{i\left(2\beta^{\frac{1}{2}}z^{\frac{1}{2}}+\frac{1}{2}\pi\alpha-\frac{3}{4}\pi\right)}\left[1+O\left(|z|^{-\frac{1}{2}}\right)\right].
\end{aligned}
$$

Comparison with (2.25), with α replaced by $(2-\alpha)$ there, shows that

$$(3.3)\qquad z^{-\frac{1}{2}\alpha+\frac{3}{4}}e^{-2i\beta^{\frac{1}{2}}z^{\frac{1}{2}}}\left[W(A_2,A_3,B_1)-k_1(\lambda)\,B_1(z,2-\alpha)\right]=O\left(|z|^{-\frac{1}{2}}\right),$$

where

$$(3.4)\qquad k_1(\lambda)=-2\pi i\,e^{\pi i\alpha}\beta^{\alpha-1}\lambda^{2\alpha}.$$

For the case when z lies in the interior of the sector S_1, by use of (3.1) we write

$$(3.5)\qquad W(A_2,A_3,B_1)=W(A_3,B_2,B_0)-W(A_1,A_3,B_2).$$

Proceding as before with the Wronskians on the right here, we find that

$$(3.6)\qquad W(A_3,B_2,B_0)=-k_1(\lambda)\,A_3(z,2-\alpha)\left[1+O\left(|z|^{-\frac{1}{2}}\right)\right]$$

and

$$(3.7)\qquad W(A_1,A_3,B_2)=-k_1(\lambda)\,B_2(z,2-\alpha)\left[1+O\left(|z|^{-\frac{1}{2}}\right)\right].$$

Since $B_1=B_2-A_3$, it follows that

$$(3.8)\qquad z^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-\xi_1}\left[W(A_2,A_3,B_1)-k_1(\lambda)\,B_1(z,2-\alpha)\right]=O\left(|z|^{-\frac{1}{2}}\right).$$

Here ξ_1 is the branch of $\xi=\frac{2}{3}i\lambda z^{\frac{3}{2}}$ which has a positive real part in S_1.

To prove (4.7), we observe that since the expression in brackets in (3.3) and (3.8) is a solution of (1.6), we may write it as a linear combination of a set of independent solutions of (1.6). Thus

$$
\begin{aligned}
(3.9)\qquad & W(A_2,A_3,B_1)-k_1(\lambda)\,B_1(z,2-\alpha)\\
&\quad= c_0B_0(z,2-\alpha)+c_1B_1(z,2-\alpha)+c_2A_2(z,2-\alpha)+c_3A_3(z,2-\alpha)
\end{aligned}
$$

for some set of quantities c_i which may depend on λ but not on z. We shall show that $c_i=0$, for $i=0,1,2,3$.

In (3.9), if we multiply through by $z^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-\xi_2}$ and let z tend to infinity along a ray in S_2, the left-hand side tends to zero because of (3.3) and the fact that ξ_2 has a positive real part in S_2. The right-hand side tends to

$$i\sqrt{\pi}\,e^{\frac{1}{2}\pi i\alpha-\frac{3}{4}\pi i}\lambda^{\alpha-\frac{3}{2}}c_3.$$

Hence we must have $c_3=0$. Similarly, if we multiply through in (3.9) by $z^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-\xi_3}$ and let z tend to infinity along a ray in S_3, we find that $c_2=0$. Then

$$(3.10)\qquad W(A_2,A_3,B_1)-k_1(\lambda)\,B_1(z,2-\alpha)=c_0B_0(z,2-\alpha)+c_1B_1(z,2-\alpha).$$

In (3.10) multiplying through by $z^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-\xi_1}$, letting z tend to infinity along a ray in S_1, and using (3.8), we find that $c_1=0$. Hence

$$(3.11)\qquad W(A_2,A_3,B_1)-k_1(\lambda)\,B_1(z,2-\alpha)=c_0B_0(z,2-\alpha).$$

If we multiply through in (3.11) by $z^{\frac{3}{4}-\frac{1}{2}\alpha} e^{-2i\beta^{\frac{1}{2}} z^{\frac{1}{2}}}$ and let z tend to infinity along a ray in $S-S_1$, the left-hand side tends to zero by virtue of (3.3). The right-hand side has the form

$$(3.12)\quad i\sqrt{\pi}\, e^{\pi i\alpha} \beta^{\frac{1}{4}-\frac{1}{2}\alpha} c_0 \left\{ e^{\frac{1}{2}\pi i\alpha - \frac{3}{4}\pi i} [1+O(|z|^{-\frac{1}{2}})] + e^{-i(4\beta^{\frac{1}{2}} z^{\frac{1}{2}} + \frac{1}{2}\pi\alpha - \frac{3}{4}\pi)} [1+O(|z|^{-\frac{1}{2}})] \right\}.$$

If the exponential function in (3.12) becomes large in magnitude as $|z|$ grows large, the quantity (3.12) tends to infinity unless $c_0=0$. On the other hand, if the exponential function tends to zero, the quantity (3.12) tends to

$$i\sqrt{\pi}\, e^{\frac{3}{2}\pi i\alpha} \beta^{\frac{1}{4}-\frac{1}{2}\alpha} e^{-\frac{3}{4}\pi i} c_0.$$

Again, we must have $c_0=0$. Since it is always possible to choose the ray in $S-S_1$ so that one of these two cases holds, we have $c_0=0$. This proves (4.7).

The relations (4.8) and (4.12) can be proved in the same manner as (4.7). Relations of the forms (3.3) and (3.8), with permutations of the subscripts, can be derived. With the assumption of relations of the type (3.9) and the appropriate changes of subscripts, it is possible to procede as in the case above.

To prove (4.1), we consider the behavior of $W(A_1, A_2, A_3)$ for large values of $|z|$. By use of (3.1), we may write

$$(3.13)\qquad W(A_1, A_2, A_3) = W(A_1, A_2, B_0) + (e^{2\pi i\alpha} - 1)\, W(A_1, A_2, B_3).$$

Then for z in $S-S_3$, with $-\frac{4}{3}\pi - \frac{2}{3}\arg\lambda < \arg z < -\frac{2}{3}\arg\lambda$, we find that the right member of (3.13) has the form

$$(3.14)\qquad \begin{aligned} -2\pi^{\frac{3}{2}} \beta^{\frac{1}{2}\alpha-\frac{3}{4}} \lambda^{2\alpha} z^{\frac{1}{2}\alpha-\frac{3}{4}} \Big\{ & e^{i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}} + \frac{1}{2}\pi\alpha - \frac{3}{4}\pi)} [1+O(|z|^{-\frac{1}{2}})] + \\ & + e^{-i(2\beta^{\frac{1}{2}} z^{\frac{1}{2}} + \frac{1}{2}\pi\alpha - \frac{3}{4}\pi)} [1+O(|z|^{-\frac{1}{2}})] \Big\}. \end{aligned}$$

Comparison with (2.10), with α replaced by $(2-\alpha)$, shows that

$$(3.15)\qquad \begin{aligned} & z^{-\frac{1}{2}\alpha+\frac{3}{4}} [W(A_1, A_2, A_3) - k_0(\lambda)\, B_0(z, 2-\alpha)] \\ & \qquad = e^{2i\beta^{\frac{1}{2}} z^{\frac{1}{2}}} O(|z|^{-\frac{1}{2}}) + e^{-2i\beta^{\frac{1}{2}} z^{\frac{1}{2}}} O(|z|^{-\frac{1}{2}}), \end{aligned}$$

where

$$(3.16)\qquad k_0(\lambda) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-2} \lambda^{2\alpha}.$$

The Wronskian $W(A_1, A_2, A_3)$ may be evaluated by similar procedures for z in the sectors $S-S_1$ and $S-S_2$, and it is found that (3.15) actually holds for all values of $\arg z$.

Since the quantity in brackets in (3.15) is a solution of (1.6), we may write

$$(3.17)\qquad \begin{aligned} & W(A_1, A_2, A_3) - k_0(\lambda)\, B_0(z, 2-\alpha) \\ & \quad = c_0 B_0(z, 2-\alpha) + c_1 B_1(z, 2-\alpha) + c_2 B_2(z, 2-\alpha) + c_3 B_3(z, 2-\alpha). \end{aligned}$$

In (3.17) we multiply through by $z^{\frac{1}{2}\alpha+\frac{1}{4}} e^{-\xi_1}$ and let z tend to infinity along a ray in S_1, then the left-hand side tends to zero, while the right-hand side tends to

$$i\sqrt{\pi}\, e^{\frac{1}{2}\pi i\alpha - \frac{3}{4}\pi i} \lambda^{\alpha-\frac{3}{2}} c_1.$$

Hence $c_1 = 0$. Similarly, if we multiply through by $z^{\frac{1}{2}\alpha+\frac{1}{4}} e^{-\xi_j}$, $j=2, 3$, and let z tend to infinity along a ray in S_j, we find that $c_j = 0$, for $j = 2, 3$. Then

$$W(A_1, A_2, A_3) - k_0(\lambda)\, B_0(z, 2-\alpha) = c_0\, B_0(z, 2-\alpha). \tag{3.18}$$

It is always possible to find a ray in S along which $e^{2i\beta^{\frac{1}{2}} z^{\frac{1}{2}}}$ becomes large in magnitude. If we multiply through in (3.18) by $z^{\frac{3}{4}-\frac{1}{2}\alpha} e^{-2i\beta^{\frac{1}{2}} z^{\frac{1}{2}}}$ and let z tend to infinity along this ray, the left-hand side tends to zero, by virtue of (3.15), while the right-hand side tends to

$$i\sqrt{\pi}\, e^{\frac{3}{2}\pi i\alpha - \frac{3}{4}\pi i} \beta^{\frac{1}{4}-\frac{1}{2}\alpha} c_0.$$

Hence $c_0 = 0$. This proves (4.1).

We have shown so far in this section that

$$\begin{aligned}
W(A_1, A_2, A_3)(z, \alpha) &= -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_0(z, 2-\alpha), \\
W(A_2, A_3, B_1)(z, \alpha) &= -2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_1(z, 2-\alpha), \\
W(A_1, A_3, B_2)(z, \alpha) &= 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \\
W(A_1, A_2, B_3)(z, \alpha) &= -2\pi i\, e^{-3\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_3(z, 2-\alpha).
\end{aligned} \tag{3.19}$$

Since the functions on the right constitute a set of linearly independent solutions for (1.6), we can determine any other third order Wronskian by the use of these formulae and the relations (3.1). For example, the Wronskian $W(A_3, B_1, B_0)$ may be determined as follows. Using (3.1), we have that

$$\begin{aligned}
W(A_3, B_1, B_0) &= W(A_3, B_1, B_3 + A_2), \\
&= W(A_3, B_1, A_2) + W(A_3, B_1, B_3), \\
&= W(A_2, A_3, B_1) + W(A_1, A_3, B_2).
\end{aligned}$$

Then from (3.19) it follows that

$$W(A_3, B_1, B_0) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} \left[B_2(z, 2-\alpha) - B_1(z, 2-\alpha)\right]$$

or

$$W(A_3, B_1, B_0) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_3(z, 2-\alpha). \tag{3.20}$$

4. Formulae for the determination of the inverse matrix

Below is given a complete list of formulae which express the third order Wronskians of the solutions of (1.1) in terms of specific solutions of (1.6). From these formulae may be calculated the elements of the inverse matrix U^{-1} corresponding to every fundamental matrix solution U of (1.2).

$$W(A_1, A_2, A_3) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_0(z, 2-\alpha), \tag{4.1}$$

$$W(A_1, A_2, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} \left[(1-e^{-2\pi i\alpha})\, B_3(z, 2-\alpha) - B_0(z, 2-\alpha)\right], \tag{4.2}$$

$$W(A_2, A_3, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} \left[(e^{2\pi i\alpha}-1)\, B_1(z, 2-\alpha) - B_0(z, 2-\alpha)\right], \tag{4.3}$$

$$W(A_1, A_3, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} \left[(1-e^{2\pi i\alpha})\, B_2(z, 2-\alpha) + B_0(z, 2-\alpha)\right], \tag{4.4}$$

$$W(A_1, A_2, B_1) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} \left[B_0(z, 2-\alpha) - B_3(z, 2-\alpha)\right], \tag{4.5}$$

$$W(A_1, A_2, B_2) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_3(z, 2-\alpha), \tag{4.6}$$

$$W(A_1, A_2, B_3) = -2\pi i\, e^{-3\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_3(z, 2-\alpha), \tag{4.7}$$

$$W(A_2, A_3, B_1) = -2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_1(z, 2-\alpha), \tag{4.8}$$

$$W(A_2, A_3, B_2) = -2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_1(z, 2-\alpha), \tag{4.9}$$

$$W(A_2, A_3, B_3) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} [B_1(z, 2-\alpha) + B_0(z, 2-\alpha)], \tag{4.10}$$

$$W(A_1, A_3, B_1) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.11}$$

$$W(A_1, A_3, B_2) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.12}$$

$$W(A_0, A_3, B_3) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.13}$$

$$W(A_1, B_2, B_3) = 0, \tag{4.14}$$

$$W(A_1, B_1, B_3) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.15}$$

$$W(A_1, B_1, B_2) = -2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.16}$$

$$W(A_2, B_1, B_3) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_2(z, 2-\alpha), \tag{4.17}$$

$$W(A_2, B_2, B_3) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_3(z, 2-\alpha), \tag{4.18}$$

$$W(A_2, B_1, B_2) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_1(z, 2-\alpha), \tag{4.19}$$

$$W(A_3, B_1, B_2) = 0, \tag{4.20}$$

$$W(A_3, B_1, B_3) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.21}$$

$$W(A_3, B_2, B_3) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.22}$$

$$W(A_1, B_1, B_0) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} [B_2(z, 2-\alpha) - B_3(z, 2-\alpha) + B_0(z, 2-\alpha)], \tag{4.23}$$

$$W(A_1, B_2, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_1(z, 2-\alpha), \tag{4.24}$$

$$W(A_1, B_3, B_0) = 2\pi i\, e^{-3\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_1(z, 2-\alpha), \tag{4.25}$$

$$W(A_2, B_1, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_2(z, 2-\alpha), \tag{4.26}$$

$$W(A_2, B_2, B_0) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} [e^{2\pi i\alpha} B_1(z, 2-\alpha) - B_3(z, 2-\alpha)], \tag{4.27}$$

$$W(A_2, B_3, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_2(z, 2-\alpha), \tag{4.28}$$

$$W(A_3, B_1, B_0) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_3(z, 2-\alpha), \tag{4.29}$$

$$W(A_3, B_2, B_0) = 2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_3(z, 2-\alpha), \tag{4.30}$$

$$\begin{aligned} W(A_3, B_3, B_0) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} [e^{2\pi i\alpha} B_2(z, 2-\alpha) - \\ - B_1(z, 2-\alpha) - B_0(z, 2-\alpha)], \end{aligned} \tag{4.31}$$

$$W(B_1, B_2, B_3) = -2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} B_2(z, 2-\alpha), \tag{4.32}$$

$$W(B_0, B_1, B_2) = -2\pi i\, e^{\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_3(z, 2-\alpha), \tag{4.33}$$

$$W(B_0, B_2, B_3) = -2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_1(z, 2-\alpha), \tag{4.34}$$

$$W(B_0, B_1, B_3) = 2\pi i\, e^{-\pi i\alpha} \beta^{\alpha-1} \lambda^{2\alpha} A_2(z, 2-\alpha). \tag{4.35}$$

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Fehlerabschätzungen bei gewöhnlichen und partiellen Differentialgleichungen

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Vorgelegt von L. COLLATZ

Es werden Fehlerabschätzungen für Funktionen $u^*(x)$ hergeleitet, welche in einem beschränkten Gebiet $\mathfrak{B}$ einer Differentialgleichung

$$M[u] = f(x, u)$$

genügen und auf dessen Rand gegebene Zusatzbedingungen

$$U_i[u] = \gamma_i \qquad (i = 1, 2, \ldots)$$

(Rand-, Anfangs- oder andere Bedingungen) erfüllen. x repräsentiert p unabhängige Veränderliche, M und die U_i bedeuten formale lineare, homogene Differentialoperatoren, zu denen es eine „Greensche Funktion" $G(x, \xi)$ mit gewissen Eigenschaften gibt. Alle vorkommenden Größen seien reell[1]. Die bewiesenen Sätze enthalten auch Existenzaussagen.

Die Näherung, deren Güte abgeschätzt werden soll, wird ermittelt, indem man einen oder mehrere Schritte des Iterationsverfahrens

$$M[u_{n+1}] = f(x, u_n), \qquad U_i[u_{n+1}] = \gamma_i \qquad (i = 1, 2, \ldots;\ n = 0, 1, 2, \ldots) \tag{0.1}$$

durchführt. Praktisch kommt man oft mit einem Iterationsschritt aus. Man beginnt dazu mit einer auf irgendeine Art erhaltenen möglichst guten Näherung $u_0(x)$ (s. Beispiel 1) oder versieht die Ausgangsnäherung $u_0(x)$ mit Parametern, die man dann so bestimmt, daß $u_1(x)$ sich von $u_0(x)$ in geeignetem Sinne möglichst wenig unterscheidet (Beispiel 2). Oft ist es auch möglich, $u_0(x)$ „rückwärts" von $u_1(x)$ ausgehend zu berechnen. Man braucht dann keine Differentialgleichung zu lösen (s. Nr. 2.4 und Beispiel 3).

Die Fehlerschranke $\sigma(x)$ für $|u^*(x) - u_1(x)|$, $u_1(x)$ bezeichnet hier die letzte nach (0.1) berechnete Näherung, erhält man im Falle $G(x, \xi) \geqq 0$ als Lösung einer „Vergleichsaufgabe"[2]

$$M[\sigma] \geqq \tilde{f}\big(x, \sigma(x) + \sigma_1(x)\big) \quad \text{mit} \quad |u_1(x) - u_0(x)| \leqq \sigma_1(x), \tag{0.2}$$

$$U_i[\sigma] = 0 \qquad (i = 1, 2, \ldots), \tag{0.3}$$

[1] Die Ergebnisse lassen sich z.T. auf komplexwertige Differentialgleichungen übertragen.

[2] Die Beziehungen (2.9), (2.10) sind von etwas allgemeinerer Form, sie enthalten — ebenso wie (1.16) — noch eine Funktion $\varrho_0(x)$. Gewöhnlich kann man jedoch $\varrho_0(x) \equiv 0$ setzen.

in der also statt einer Differentialgleichung eine Differentialungleichung vorkommt. Auch die Zusatzbedingungen (0.3) lassen sich in vielen Fällen durch Ungleichungen ersetzen (s. die Nrn. 3.2 und 4.2). Nimmt $G(x,\xi)$ auch negative Werte an, muß man an Stelle der Beziehungen (0.2), (0.3) die Integralungleichung (1.16) verwenden.

Die Funktion $\tilde{f}(x,y)$ ist eine Majorante der Funktion $f(x,y)$ (im Sinne der Beziehungen (1.5)) mit $\tilde{f}(x,0)\equiv 0$. Sie kann in y linear oder nichtlinear sein. Die geeignete Wahl von $\tilde{f}(x,y)$ wird in Nr. 2.3 und in den Beispielen der §§ 3 und 4 erläutert. Im Spezialfall des in Nr. 2.4 behandelten vereinfachten Newtonschen Verfahrens ist es z.B. besonders zweckmäßig, eine in y nichtlineare Majorante zu benutzen.

Bei der Herleitung der Ergebnisse stützen wir uns auf Sätze über das Iterationsverfahren $u_{n+1}=T u_n$ in abstrakten Räumen, welche in [*11*] als Spezialfall (und vorher in etwas anderer Form in [*10*]) bewiesen wurden[3]. Für bestimmte Aufgaben bei gewöhnlichen Differentialgleichungen wurden Fehlerabschätzungen ähnlicher Art mit in y linearer Majorante $\tilde{f}(x,y)$ bereits in [*8*] mitgeteilt. Allerdings wurde dort eine zusätzliche, hier nicht erforderliche Konvergenzbedingung benötigt. Die Fehlerschranken, welche der bekannte Fixpunktsatz für kontrahierende Abbildungen [*12*], [*3*] liefert, sind in den hier für lineare Operatoren hergeleiteten enthalten (s. Nr. 2.3).

In § 1 werden die Sätze aus [*11*] auf Integralgleichungen angewendet, in § 2 die dabei gewonnenen Ergebnisse auf Differentialgleichungen übertragen. Die §§ 3 und 4 behandeln spezielle einfache Klassen von Rand- und Anfangswertaufgaben bei gewöhnlichen und partiellen Differentialgleichungen und bringen numerische Beispiele.

Zur Rechnung der numerischen Beispiele wurde die am Institut für Angewandte Mathematik der Universität Hamburg aufgestellte Rechenanlage IBM 650 benutzt (s. insbesondere Beispiel 1). Eine Vereinigung von funktional-analytischer Theorie und der Verwendung elektronischer Anlagen für die praktische Rechnung scheint in der numerischen Analysis Erfolg zu versprechen. Zum Beispiel wäre es im Sinne dieser Arbeit sehr erwünscht, Verfahren für Rechenanlagen zu entwickeln, mit denen man in einer mit Parametern versehenen Funktion F diese Parameter so bestimmen kann, daß F eine gegebene Funktion von einer Seite her möglichst gut annähert, denn auf eine solche Aufgabe führt die Ermittelung einer Schranke $\sigma(x)$ aus (0.2), (0.3).

§ 1. Integralgleichungen

1.1. Aufgabenstellung. $\mathfrak{B}$ sei ein (nicht notwendig offenes) beschränktes Gebiet des p-dimensionalen Euklidischen Raumes. Die Punkte dieses Raumes bezeichnen wir mit x oder ξ; y und z bedeuten dagegen reelle Zahlen. Aussagen über Funktionen von x sollen auf dem durch Abschließen von $\mathfrak{B}$ entstehenden Gebiet $\overline{\mathfrak{B}}$ gelten. Gesucht sind stetige Funktionen $u(x)$, welche einer gegebenen Integralgleichung

$$u(x)=g(x)+\int_{\mathfrak{B}} G(x,\xi)\, f\big(\xi,u(\xi)\big)\, d\xi$$

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[3] Die dortigen Ergebnisse enthalten auch Eindeutigkeitsaussagen. Auf solche verzichten wir hier.

genügen. Die Funktionen $g(x)$, $G(x,\xi)$ und $f(x,y)$ mögen dabei folgende Eigenschaften besitzen.

a) $g(x)$ sei stetig.

b) $G(x,\xi)$ sei auf $\overline{\mathfrak{B}}\times\overline{\mathfrak{B}}$ definiert, abgesehen höchstens von Argumentepaaren x,ξ mit $x=\xi$. Zu jedem $\varepsilon>0$ gebe es eine Zahl $\delta(\varepsilon)$ derart, daß

$$\int_{\mathfrak{B}} |G(x,\xi)-G(x',\xi)|\,d\xi<\varepsilon \quad \text{für} \quad x,x'\in\overline{\mathfrak{B}} \quad \text{mit} \quad |x-x'|<\delta \tag{1.1}$$

gilt, wobei $|x-x'|$ den Euklidischen Abstand der Punkte x und x' bedeutet.

Bei beliebigen stetigen Funktionen $r(x)$ und $s(x)$ mit $|r(x)|\leqq s(x)$ gelte

$$\left|\int_{\mathfrak{B}} K\,r(\xi)\,d\xi\right|\leqq\int_{\mathfrak{B}}|K|\,|r(\xi)|\,d\xi\leqq\int_{\mathfrak{B}}|K|\,s(\xi)\,d\xi, \tag{1.2}$$

wenn K die Funktion $G(x,\xi)$ oder eine Differenz $G(x,\xi)-G(x',\xi)$ $(x,x'\in\overline{\mathfrak{B}})$ ist. Dies bedeutet gleichzeitig, daß die auftretenden Integrale (im Sinne irgendeiner der üblichen Definitionen) existieren sollen.

Die Funktion $|G(x,\xi)|$ erfülle ebenfalls die von $G(x,\xi)$ geforderten Voraussetzungen.

c) Die Funktion $f(x,y)$ sei stetig auf einem $\big((p+1)$-dimensionalen$\big)$ Gebiet[4]

$$\mathfrak{G}:\quad x\in\overline{\mathfrak{B}},\quad \varphi(x)\leqq y\leqq\psi(x)\qquad \big(\varphi(x),\psi(x) \text{ auf } \overline{\mathfrak{B}} \text{ definiert}\big). \tag{1.3}$$

Zu einer geeignet gewählten, in $\mathfrak{G}$ verlaufenden stetigen Funktion $w(x)$ gebe es eine auf einem Gebiet[4]

$$\tilde{\mathfrak{G}}:\quad x\in\overline{\mathfrak{B}},\quad 0\leqq y\leqq\vartheta(x)\qquad \big(\vartheta(x) \text{ auf } \overline{\mathfrak{B}} \text{ definiert}\big) \tag{1.4}$$

erklärte stetige Funktion $\tilde{f}(x,y)$ derart, daß

$$\begin{aligned}&\tilde{f}(x,0)\equiv 0,\\ &0\leqq\tilde{f}(x,y+z)-\tilde{f}(x,y)\leqq\tilde{f}(x,y'+z')-\tilde{f}(x,y')\quad\text{für}\quad 0\leqq y\leqq y',\ 0\leqq z\leqq z',\\ &|f(x,y)-f(x,z)|\leqq\tilde{f}\big(x,|y-z|+|z-w(x)|\big)-\tilde{f}\big(x,|z-w(x)|\big)\end{aligned} \tag{1.5}$$

gilt, soweit die jeweils vorkommenden Argumentepaare in $\mathfrak{G}$ bzw. $\tilde{\mathfrak{G}}$ liegen.

Die gegebene Integralgleichung soll mit dem Iterationsverfahren

$$u_{n+1}(x)=g(x)+\int_{\mathfrak{B}} G(x,\xi)\,f\big(\xi,u_n(\xi)\big)\,d\xi\qquad (n=0,1,2,\ldots) \tag{1.6}$$

gelöst werden. Dabei sei $u_0(x)$ stetig und verlaufe in $\mathfrak{G}$.

Die Voraussetzungen lassen sich mildern: $g(x)$ darf unstetig sein. Ist dies der Fall, so sind Lösungen der Form

$$u(x)=g(x)+\hat{u}(x) \tag{1.7}$$

[4] Die beiden Zeichen $\leqq$ in (1.3) und das rechte in (1.4) können z.T. oder insgesamt durch $<$ ersetzt werden; auch die Fälle $\varphi=-\infty$, $\psi=+\infty$, $\vartheta=+\infty$ sind zugelassen. Man wird sogar im allgemeinen $\vartheta=+\infty$ setzen. In entsprechender Weise sind dann die weiter vorkommenden Ungleichungen abzuändern, welche φ, ψ oder ϑ enthalten.

mit stetigem $\hat{u}(x)$ gesucht. Die Ungleichungen (1.2) mögen dann auch für alle Funktionen r und s mit $|r| \leqq s$ gelten, welche sich in der Gestalt $f(x, u)$ mit einer Funktion u der Art (1.7) schreiben lassen, und w und u_0 sollen dann ebenfalls die Gestalt (1.7) haben: $w = g + \hat{w}$, $u_0 = g + \hat{u}_0$.

1.2. Beispiele für Majoranten $\tilde{f}(x, y)$. $\tilde{f}(x, y)$ bezeichnen wir als Majorante der Funktion $f(x, y)$ bezüglich $w(x)$. Im allgemeinen wird man $w(x) = u_0(x)$ wählen. Meistens läßt sich $\vartheta(x) = \infty$ setzen. Im folgenden werden Beispiele für solche Majoranten angegeben. Wir nennen $f(x, y)$ oder $\tilde{f}(x, y)$ linear, falls diese Funktionen in y linear sind, andernfalls nichtlinear. Eine lineare Funktion $\tilde{f}(x, y)$ ist auch homogen in y, bei $f(x, y)$ braucht dies nicht der Fall zu sein.

Es sei nun $f(x, y)$ eine auf einem Gebiet $\mathfrak{G}$ (1.3) erklärte stetige Funktion und $w(x)$ eine beliebige in $\mathfrak{G}$ verlaufende stetige Funktion.

Hilfssatz 1. *Die Ableitung $f_y(x, y)$ existiere und sei stetig auf $\mathfrak{G}$. Dann erfüllt eine auf einem Gebiet $\tilde{\mathfrak{G}}$ (1.4) erklärte stetige Funktion $\tilde{f}(x, y)$ die Voraussetzungen (1.5), wenn sie eine stetige, nichtnegative, mit y monoton nichtfallende Ableitung $\tilde{f}_y(x, y)$ besitzt und*

$$\tilde{f}(x, 0) \equiv 0 \tag{1.8}$$

sowie

$$|f_y(x, y + w(x))| \leqq \tilde{f}_y(x, |y|)$$

gilt, soweit die Punkte $x, y + w(x)$ und $x, |y|$ in $\mathfrak{G}$ bzw. $\tilde{\mathfrak{G}}$ liegen.

Beweis. Es ist (soweit die vorkommenden Funktionswerte erklärt sind)

$$\tilde{f}(x, y + z) - \tilde{f}(x, y) = \int_0^1 \tilde{f}_y(x, y + t z)\, z\, dt \geqq 0$$

und

$$\int_0^1 \tilde{f}_y(x, y + t z)\, z\, dt \leqq \int_0^1 \tilde{f}_y(x, y' + t z')\, z'\, dt = \tilde{f}(x, y' + z') - \tilde{f}(x, y')$$

$$\text{für} \quad 0 \leqq y \leqq y',\ 0 \leqq z \leqq z'.$$

Ferner erhält man

$$\begin{aligned}
|f(x, y) - f(x, z)| &= \left|\int_0^1 f_y\big(x, z + t(y - z)\big)\,(y - z)\, dt\right| \\
&\leqq \int_0^1 |f_y(x, z - w + t(y - z) + w)|\, |y - z|\, dt \\
&\leqq \int_0^1 \tilde{f}_y\big(x, |z - w + t(y - z)|\big)\, |y - z|\, dt \\
&\leqq \int_0^1 \tilde{f}_y\big(x, |z - w| + t|y - z|\big)\, |y - z|\, dt \\
&= \tilde{f}\big(x, |y - z| + |z - w|\big) - \tilde{f}\big(x, |z - w|\big).
\end{aligned}$$

Aus Hilfssatz 1 folgert man leicht den

Hilfssatz 2. *Die Funktion $f(x, y)$ sei für $x \in \overline{\mathfrak{B}}$, $-\infty < y < \infty$ erklärt und besitze in diesem Gebiet eine stetige Ableitung $f_y(x, y)$. Gilt*

$$|f_y(x, -y + w(x))| \leqq f_y\big(x, y + w(x)\big) \leqq f_y\big(x, y' + w(x)\big) \quad \text{für} \quad 0 \leqq y \leqq y' < \infty, \tag{1.9}$$

so ist

$$\tilde{f}(x, y) = f(x, y + w(x)) - f(x, w(x)) \tag{1.10}$$

eine für $x \in \overline{\mathfrak{B}}$, $0 \leqq y < \infty$ *definierte Majorante der Funktion* $f(x, y)$ *bezüglich* $w(x)$.

Polynome in y als Majoranten liefert der

Hilfssatz 3. *Die Funktion* $f(x, y)$ *besitze auf* $\mathfrak{G}$ *stetige Ableitungen*

$$f^{(i)}(x, y) = \frac{\partial^i f}{\partial y^i}(x, y) \qquad (i = 0, 1, 2, \ldots, m-1;\ m \geqq 1),$$

und es gebe stetige Funktionen $a_i(x)$, *mit denen*

$$|f^{(i)}(x, w(x))| \leqq a_i(x) \qquad (i = 1, 2, \ldots, m-1),$$

$$|f^{(m-1)}(x, y) - f^{(m-1)}(x, z)| \leqq a_m(x)\,|y - z| \quad \text{für} \quad x \in \overline{\mathfrak{B}},\ \varphi(x) \leqq y, z \leqq \psi(x) \tag{1.11}$$

gilt. Dann ist die für $x \in \overline{\mathfrak{B}}$, $0 \leqq y < \infty$ *erklärte Funktion*

$$\tilde{f}(x, y) = \sum_{i=1}^{m} \frac{1}{i!}\, a_i(x)\, y^i$$

eine Majorante der Funktion $f(x, y)$ *bezüglich* $w(x)$.

Gilt z.B. für die Funktion $f(x, y)$ selbst eine Lipschitz-Bedingung

$$|f(x, y) - f(x, z)| \leqq a_1(x)\,|y - z|, \tag{1.12}$$

so kann man die lineare Majorante $\tilde{f}(x, y) = a_1(x)\, y$ verwenden.

Beweis des Hilfssatzes 3. Für $m = 1$ ist die Richtigkeit der Behauptung leicht einzusehen. Im folgenden sei $m \geqq 2$. Man berechnet

$$f(x, y) - f(x, z) = \sum_{i=1}^{m-1} \frac{1}{i!} f^{(i)}(x, w(x))\,[(y - w(x))^i - (z - w(x))^i] + R \tag{1.13}$$

mit

$$R = \int_0^1 \int_0^1 \int_0^{t_{m-2}} \cdots \int_0^{t_3} \int_0^{t_2} [f^{(m-1)}(x, w + t_1(\eta - w)) - f^{(m-1)}(x, w)] \times$$
$$\times (\eta - w)^{m-2} (y - z)\, dt_1\, dt_2 \ldots dt_{m-1}$$

und $\eta = z + t_{m-1}(y - z)$, indem man die Integrationen ausführt. Ferner schätzt man ab:

$$|(y - w) - (z - w)|^i = |[(y - z) + (z - w)]^i - (z - w)^i| \leqq (|y - z| + |z - w|)^i - |z - w|^i,$$

sowie mit (1.11):

$$|R| \leqq \int_0^1 \int_0^1 \int_0^{t_{m-2}} \cdots \int_0^{t_2} a_m(x)\, t_1\, |\eta - w|^{m-1} |y - z|\, dt_1 \ldots dt_{m-1}$$

$$= \frac{a_m(x)}{(m-1)!} \int_0^1 |z - w + t_{m-1}(y - z)|^{m-1} |y - z|\, dt_{m-1}$$

$$\leqq \frac{a_m(x)}{(m-1)!} \int_0^1 (|z - w| + t_{m-1}|y - z|)^{m-1} |y - z|\, dt_{m-1}$$

$$= \frac{a_m(x)}{m!} [(|y - z| + |z - w|)^m - |z - w|^m].$$

Insgesamt ergibt sich dann aus (1.13)

$$|f(x,y)-f(x,z)| \leqq \sum_{i=1}^{m} \frac{1}{i!} a_i(x) \left[(|y-z|+|z-w(x)|)^i - |z-w(x)|^i\right].$$

1.3. Funktionalanalytische Formulierung der Aufgabe. $\mathfrak{R}$ bedeute die Menge der Funktionen $u(x)$, welche sich in der Form

$$u(x) = g(x) + \hat{u}(x)$$

mit auf $\overline{\mathfrak{B}}$ stetiger Funktion $\hat{u}(x)$ schreiben lassen. (Bei stetigem $g(x)$ ist $\mathfrak{R}$ also die Menge der auf $\overline{\mathfrak{B}}$ stetigen Funktionen. Nur unter den in Nr. 1.1 genannten schwächeren Voraussetzungen über $g(x)$ muß man $u(x)$ in dieser Form darstellen.) Für jedes Elementepaar $u, v \in \mathfrak{R}$ wird durch

$$\delta(u,v) = |u(x) - v(x)| \qquad (-|\hat{u}(x) - \hat{v}(x)|)$$

ein Abstand $\delta(u,v)$ definiert. Diesen fassen wir als Element des Raumes $\mathfrak{N}$ der auf $\overline{\mathfrak{B}}$ stetigen Funktionen auf. Für die Elemente $\varrho, \sigma, \ldots$ des Raumes $\mathfrak{N}$ werden die Addition $\varrho+\sigma$, die skalare Multiplikation $\alpha\varrho$ und die Beziehung $\varrho \leqq \sigma$ in der bei Funktionen üblichen Weise erklärt. Sind ϱ und ϱ_n $(n=1, 2, \ldots)$ Elemente $\in \mathfrak{N}$, so bedeute $\varrho = \lim \varrho_n$, daß die Funktionenfolge $\varrho_n(x)$ für $n \to \infty$ gleichmäßig auf $\overline{\mathfrak{B}}$ gegen $\varrho(x)$ konvergiert.

In $\mathfrak{R}$ wird ein Grenzwertbegriff für Folgen u_n $(n=1, 2, \ldots)$ dadurch definiert, daß $u = \lim u_n$ der Beziehung $\lim \delta(u, u_n) = 0$ äquivalent sein soll. Dieser Limes-Begriff bedeutet ebenfalls die auf $\overline{\mathfrak{B}}$ gleichmäßige Konvergenz. $\mathfrak{R}$ ist vollständig in dem Sinne, daß jede Folge $u_n(x)$, für die $\lim \delta(u_{k_n}, u_n) = 0$ bei beliebiger monotoner Folge k_n gilt, ein Grenzelement $u \in \mathfrak{R}$ besitzt $(\lim \delta(u, u_n) = 0)$.

Es sei nun $\mathfrak{D}$ die Menge der Funktionen $u \in \mathfrak{R}$, für welche die Punkte $x, u(x)$ in $\mathfrak{G}$ liegen, und T der auf $\mathfrak{D}$ durch

$$Tu = g(x) + \int_{\mathfrak{B}} G(x,\xi)\, f(\xi, u(\xi))\, d\xi$$

definierte Operator. Wegen der von $G(x,\xi)$ verlangten Eigenschaften ist das Integral eine stetige Funktion von x, T bildet also $\mathfrak{D}$ in $\mathfrak{R}$ ab. Die Gleichung $u = Tu$ ist dem Ausgangsproblem äquivalent, und (1.6) läßt sich in der Form

$$u_{n+1} = T u_n \qquad (n = 0, 1, 2, \ldots), \qquad u_0 \in \mathfrak{D} \tag{1.14}$$

schreiben.

1.4. Abschätzen des Operators T. Wir wollen einen in [*11*] bewiesenen Satz benutzen, um Aussagen über das Iterationsverfahren (1.14) zu erhalten. Zu diesem Zwecke schätzen wir den Abstand $\delta(Tu, Tv)$ ab. Wegen (1.5) ist

$$\begin{aligned}
\delta(Tu, Tv) &= |Tu - Tv| \\
&= \left|\int_{\mathfrak{B}} G(x,\xi)\,[f(\xi,u) - f(\xi,v)]\, d\xi\right| \leqq \int_{\mathfrak{B}} |G(x,\xi)|\, |f(\xi,u) - f(\xi,v)|\, d\xi \\
&\leqq \int_{\mathfrak{B}} |G(x,\xi)|\, [\tilde{f}(\xi, |u-v| + |v-w|) - \tilde{f}(\xi, |v-w|)]\, d\xi.
\end{aligned}$$

Das ist eine Ungleichung der in [11] als Spezialfall behandelten Form

$$\delta(Tu, Tv) \leqq F[\delta(u,v), \delta(u,w), \delta(v,w)]$$

mit

$$F[\varrho^1, \varrho^2, \varrho^3] = P(\varrho^1 + \varrho^3) - P\varrho^3$$

und dem auf der Menge $\tilde{\mathfrak{D}}$ der in $\tilde{\mathfrak{G}}$ verlaufenden stetigen Funktionen $\varrho(x)$ durch

$$P\varrho = \int_{\mathfrak{B}} |G(x,\xi)|\, \tilde{f}(\xi, \varrho(\xi))\, d\xi$$

definierten Operator.

1.5. Untersuchen des Vergleichsverfahrens. Konvergenzaussagen und Fehlerabschätzungen für das Verfahren (1.14) erhalten wir nach [11], § 1 mit Hilfe eines Vergleichsverfahrens

$$\varrho_{n+1} = \tilde{T}\varrho_n \qquad (n = 0, 1, 2, \ldots). \tag{1.15}$$

$\tilde{T}$ bedeutet dabei den durch

$$\tilde{T}\varrho = \sigma_1 + \varrho_0 + P\varrho - P\varrho_0$$

aus $\tilde{\mathfrak{D}}$ definierten Operator; ϱ_0 und σ_1 sind stetige Funktionen mit

$$|u_0(x) - w(x)| \leqq \varrho_0(x), \quad |u_1(x) - u_0(x)| \leqq \sigma_1(x), \quad \varrho_0(x) + \sigma_1(x) \leqq \vartheta(x)\,[5].$$

Wir untersuchen die Konvergenz dieses Vergleichsverfahrens. Gibt es eine stetige Funktion $\tau(x)$ mit $\varrho_0(x) \leqq \tau(x) \leqq \vartheta(x)$, welche der Ungleichung

$$\tau \geqq \tilde{T}\tau$$

genügt, so ist die Iteration (1.15) nach Zusatz 1 in [11] unbeschränkt durchführbar, die Folge ϱ_n nimmt monoton nicht ab und ist durch τ nach oben beschränkt. Die Folge $\varrho_n(x)$ konvergiert also punktweise gegen eine auf $\overline{\mathfrak{B}}$ erklärte Funktion $\varrho^*(x)$.

Wir beweisen, daß die Konvergenz auf $\overline{\mathfrak{B}}$ gleichmäßig ist. Die Funktionen $\varrho_n(x)$ $(n = 1, 2, \ldots)$ sind gleichmäßig beschränkt und gleichgradig stetig, denn es gilt

$$0 \leqq \varrho_n(x) \leqq \tau(x) \leqq \operatorname*{Max}_{\overline{\mathfrak{B}}} \tau(x)$$

und infolge (1.1) mit der stetigen Funktion $\chi(x) = \sigma_1 + \varrho_0 - P\varrho_0$ bei gegebenem $\varepsilon > 0$

$$\begin{aligned} |\varrho_n(x) - \varrho_n(x')| &\leqq |\chi(x) - \chi(x')| + \int_{\mathfrak{B}} |G(x,\xi) - G(x',\xi)|\, \tilde{f}(\xi, \varrho_{n-1})\, d\xi \\ &\leqq |\chi(x) - \chi(x')| + \int_{\mathfrak{B}} |G(x,\xi) - G(x',\xi)|\, \tilde{f}(\xi, \tau)\, d\xi \\ &\leqq |\chi(x) - \chi(x')| + \int_{\mathfrak{B}} |G(x,\xi) - G(x',\xi)|\, d\xi \cdot \operatorname*{Max}_{\overline{\mathfrak{B}}} \tilde{f}(x, \tau(x)) < \varepsilon \end{aligned}$$

für genügend kleine Werte $|x - x'|$. Die Folge $\varrho_n(x)$ enthält also eine gleichmäßig konvergente Teilfolge und konvergiert damit selbst gleichmäßig, da sie monoton ist. $\varrho^*(x)$ ist also stetig und es gilt $\varrho^* = \lim \varrho_n$ im Sinne unserer Definition.

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[5] Die Möglichkeit einer solchen Abschätzung mit $\varrho_0 + \sigma_1 \leqq \vartheta$ wird vorausgesetzt. Siehe Fußnote 6.

ϱ^* erfüllt die Gleichung $\varrho^* = \tilde{T}\varrho^*$, denn wegen der gleichmäßigen Konvergenz der Folge $\varrho_n(x)$ gegen $\varrho^*(x)$ und der Stetigkeit der Funktion $\tilde{f}(x, y)$ ist bei gegebenem $\varepsilon > 0$

$$\begin{aligned}|\varrho^* - \tilde{T}\varrho^*| &\leqq |\varrho^* - \varrho_n| + |\tilde{T}\varrho^* - \tilde{T}\varrho_{n-1}| \\ &= |\varrho^* - \varrho_n| + \Big|\int_{\mathfrak{B}} G(x,\xi)\,[\tilde{f}(\xi,\varrho^*) - \tilde{f}(\xi,\varrho_{n-1})]\,d\xi\Big| \\ &\leqq |\varrho^*(x) - \varrho_n(x)| + \operatorname{Max}_{\overline{\mathfrak{B}}} \int_{\mathfrak{B}} |G(x,\xi)|\,d\xi \cdot \operatorname{Max}_{\overline{\mathfrak{B}}} |\tilde{f}(x,\varrho^*(x)) - \tilde{f}(x,\varrho_{n-1}(x))| < \varepsilon\end{aligned}$$

für genügend großes n. Das hierbei auftretende Maximum der Funktion $\int_{\mathfrak{B}} |G(x,\xi)|\,d\xi$ existiert, da diese Funktion infolge der von $G(x,\xi)$ verlangten Eigenschaften stetig ist.

Praktisch wird man im allgemeinen nicht τ ermitteln, sondern $\sigma = \tau - \sigma_1 - \varrho_0$. Diese Funktion muß dann der Ungleichung

$$\sigma \geqq P(\sigma + \sigma_1 + \varrho_0) - P\varrho_0$$

genügen.

1.6. Ergebnis. Mit Satz 1 aus [*11*] erhält man unter den in Nr. 1.1 genannten Voraussetzungen das folgende Ergebnis für das Iterationsverfahren (1.6) zur Lösung der gegebenen Aufgabe.

Satz 1. *$\varrho_0(x)$ und $\sigma_1(x)$ seien stetige Funktionen, welche die Bedingungen*

$$|u_0(x) - w(x)| \leqq \varrho_0(x), \quad |u_1(x) - u_0(x)| \leqq \sigma_1(x), \quad \varrho_0(x) + \sigma_1(x) \leqq \vartheta(x)\,^6$$

erfüllen. Gibt es eine stetige Funktion $\sigma(x)$ mit

$$0 \leqq \sigma(x) \quad \textit{und} \quad \sigma(x) \leqq \vartheta(x) - \varrho_0(x) - \sigma_1(x)\,^6,$$

welche der Ungleichung

$$\sigma(x) \geqq \int_{\mathfrak{B}} |G(x,\xi)|\,[\tilde{f}(\xi,\sigma(\xi) + \sigma_1(\xi) + \varrho_0(\xi)) - \tilde{f}(\xi,\varrho_0(\xi))]\,d\xi \tag{1.16}$$

genügt und gilt

$$\varphi(x) + \sigma(x) \leqq u_1(x) \leqq \psi(x) - \sigma(x), \tag{1.17}$$

so ist durch die Iterationsvorschrift (1.6) *eine Folge von Funktionen $u_n(x)$ $(n = 0, 1, 2, \ldots)$ definiert, welche gleichmäßig auf $\overline{\mathfrak{B}}$ gegen eine Lösung $u^*(x)$ der gegebenen Aufgabe konvergiert, und es gilt die Fehlerabschätzung*

$$|u^*(x) - u_1(x)| \leqq \sigma(x), \tag{1.18}$$

sowie allgemeiner

$$|u^*(x) - u_n(x)| \leqq \tau(x) - \varrho_n(x) \qquad (n = 0, 1, 2, \ldots)$$

mit $\tau = \sigma + \sigma_1 + \varrho_0$ und den durch (1.15) *definierten Funktionen $\varrho_n(x)$.*

Die Voraussetzung (1.17) hat zur Folge, daß $\mathfrak{D}$ alle Funktionen $u \in \mathfrak{R}$ mit

$$|u(x) - u_1(x)| \leqq \sigma(x) \tag{1.19}$$

enthält. Nach Fußnote 5 in [*11*] braucht $\mathfrak{D}$ aber sogar nur die (1.19) genügenden Funktionen $u \in T\mathfrak{D}$ und deren Häufungselemente zu umfassen. Deshalb gilt der

[6] Im Normalfall $\vartheta = \infty$ ist diese Ungleichung immer erfüllt.

Zusatz 1.1. *Die Forderung* (1.17) *läßt sich durch*

$$\operatorname{Max}\left[\Phi(x), u_1(x) - \sigma(x)\right] \geqq \varphi(x), \quad \operatorname{Min}\left[\Psi(x), u_1(x) + \sigma(x)\right] \leqq \psi(x) \tag{1.20}$$

ersetzen, wenn $\Phi(x)$ *und* $\Psi(x)$ *irgendwelche Funktionen bedeuten, mit denen*

$$\Phi(x) \leqq g(x) + \int_{\mathfrak{B}} G(x,\xi)\, f(\xi, y)\, d\xi \leqq \Psi(x) \tag{1.21}$$

für

$$x \in \overline{\mathfrak{B}}, \quad \varphi(x) \leqq y \leqq \psi(x)$$

gilt.

(1.20) ist z.B. immer erfüllt, falls $\varphi(x) \leqq \Phi(x) \leqq \Psi(x) \leqq \psi(x)$ ist.

Mit dem folgenden Zusatz 1.2 kann man einfache Schranken $\sigma(x)$ gewinnen, wenn man (1.16) nicht direkt lösen will. In diesem Zusatz benutzen wir weitere

Voraussetzungen: Die Funktion $\tilde{f}(x, y)$ sei für $x \in \overline{\mathfrak{B}}$, $0 \leqq y < \infty$ erklärt. Es gebe stetige, nichtnegative Funktionen $V(x)$ und $W(x)$, eine für $\alpha \geqq 0$ definierte stetige Funktion $p(\alpha)$ sowie Konstanten α_0, α_1 und $\gamma > 0$ mit folgenden Eigenschaften:

$$p(0) = 0,$$

$$\tilde{f}\big(x, (\alpha+\beta)\, W(x)\big) - \tilde{f}\big(x, \alpha\, W(x)\big) \leqq \left[p(\alpha+\beta) - p(\alpha)\right] V(x) \quad \text{für} \quad \alpha, \beta \geqq 0, \tag{1.22}$$

$$|u_0(x) - w(x)| \leqq \alpha_0 W(x), \quad |u_1(x) - u_0(x)| \leqq \alpha_1 W(x),$$

$$\int_{\mathfrak{B}} |G(x,\xi)|\, V(\xi)\, d\xi \leqq \gamma\, W(x). \tag{1.23}$$

Zusatz 1.2 *Gibt es unter den obigen Voraussetzungen eine Zahl* $\eta \geqq 0$ *mit*

$$\eta \geqq \gamma\left[p(\eta + \alpha_1 + \alpha_0) - p(\alpha_0)\right],$$

so genügt die Funktion

$$\sigma(x) = \frac{\eta}{\gamma} \int_{\mathfrak{B}} |G(x,\xi)|\, V(\xi)\, d\xi \tag{1.24}$$

(*und ebenso*

$$\sigma(x) = \eta\, W(x)) \tag{1.25}$$

der Ungleichung (1.16) *bei* $\varrho_0 = \alpha_0 W(x)$ *und* $\sigma_1 = \alpha_1 W(x)$.

Die Richtigkeit dieser Behauptung folgt mit $\sigma \leqq \eta W$ aus

$$\begin{aligned}\int |G|\left[\tilde{f}(\xi, \sigma + \sigma_1 + \varrho_0) - \tilde{f}(\xi, \varrho_0)\right] d\xi &\leqq \int |G|\left[\tilde{f}(\xi, (\eta + \alpha_1 + \alpha_0)\, W) - \tilde{f}(\xi, \alpha_0 W)\right] d\xi \\ &\leqq \left[p(\eta + \alpha_1 + \alpha_0) - p(\alpha_0)\right] \int |G|\, V\, d\xi \leqq \frac{\eta}{\gamma} \int |G|\, V\, d\xi \leqq \eta\, W.\end{aligned}$$

Bei linearer Majorante $\tilde{f}(x, y) = a_1(x)\, y$ benutzt man im Zusatz 1.2 zweckmäßigerweise $p(\alpha) = \alpha$. Die Ungleichung (1.22) erhält dann die einfache Form

$$a_1(x)\, W(x) \leqq V(x),$$

und unter der Voraussetzung $\gamma < 1$ ergibt sich $\eta = \frac{\alpha_1 \gamma}{1-\gamma}$, so daß dann die Schranken (1.24) und (1.25) in

$$\sigma(x) = \frac{\alpha_1}{1-\gamma} \int_{\mathfrak{B}} |G(x,\xi)|\, V(\xi)\, d\xi \tag{1.26}$$

$$\left(\text{bzw.} \quad \sigma(x) = \frac{\alpha_1 \gamma}{1-\gamma}\, W(x)\right) \tag{1.27}$$

übergehen.

§ 2. Differentialgleichungen

2.1. Aufgabenstellung. Wie in Nr. 1.1 sei $\mathfrak{B}$ ein (nicht notwendig offenes) beschränktes Gebiet des p-dimensionalen Raumes. Sein Rand werde mit Γ bezeichnet. Aussagen über Funktionen von x sollen auch hier auf $\overline{\mathfrak{B}} = \mathfrak{B} + \Gamma$ gelten, sofern nichts anderes gesagt wird. Gesucht ist die Lösung $u^*(x)$ einer gegebenen Differentialgleichung

$$M[u] = f(x, u), \tag{2.1}$$

welche auf Γ (oder Teilen von Γ) gegebene Randbedingungen

$$U_i[u] = \gamma_i \qquad (i = 1, 2, \ldots) \tag{2.2}$$

erfüllt. M und die U_i seien (formale) lineare, homogene Differentialoperatoren. Unter „Randbedingungen" verstehen wir dabei im erweiterten Sinne irgendwelche „Bedingungen auf dem Rande". Damit werden dann unter anderem auch Anfangswertaufgaben erfaßt (s. § 4).

Als Lösung einer Aufgabe

$$M[u] = b \qquad \bigl(\text{oder auch } M[u] \geqq b\bigr), \tag{2.3}$$

$$U_i[u] = \gamma_i \qquad (i = 1, 2, \ldots) \tag{2.4}$$

oder

$$U_i[u] = 0 \qquad (i = 1, 2, \ldots) \tag{2.5}$$

bei gegebener Funktion $b(x)$ oder $b(x, u)$ (oder einer anderen $M[u]$ und die $U_i[u]$ enthaltenden Aufgabe) bezeichnen wir eine stetige Funktion mit bestimmten von der speziellen Aufgabenstellung abhängenden Differenzierbarkeitseigenschaften, welche auf $\mathfrak{B}$ der Differentialgleichung (bzw. -ungleichung) (2.3) genügt und auf Γ die Randbedingungen (2.4) bzw. (2.5) erfüllt.

Die Aufgabe

$$M[u] = 0, \qquad U_i[u] = \gamma_i \qquad (i = 1, 2, \ldots)$$

besitze eine Lösung $g(x)$. Ferner existiere eine „Greensche Funktion" $G(x, \xi)$ mit folgenden Eigenschaften:

a) $G(x, \xi)$ erfülle die in Nr. 1.1 genannten Voraussetzungen b).

b) Ist die Aufgabe

$$M[u] = r(x), \qquad U_i[u] = \gamma_i \qquad (i = 1, 2, \ldots)$$

für eine gegebene stetige Funktion $r(x)$ lösbar, so habe die Lösung die Gestalt

$$u(x) = g(x) + \int_{\mathfrak{B}} G(x, \xi)\, r(\xi)\, d\xi.$$

c) Das Ausgangsproblem (2.1), (2.2) sei der Aufgabe äquivalent, eine stetige Lösung der Integralgleichung

$$u(x) = g(x) + \int_{\mathfrak{B}} G(x, \xi)\, f\bigl(\xi, u(\xi)\bigr)\, d\xi \tag{2.6}$$

zu ermitteln.

d) Die Funktion $f(x, y)$ habe die in Nr. 1.1 unter c) genannten Eigenschaften (bezüglich der geeigneten Wahl des Gebietes $\mathfrak{G}$ s. Nr. 2.3).

Zur Lösung der Aufgabe (2.1), (2.2) wird das Iterationsverfahren

$$M[u_{n+1}] = f(x, u_n), \qquad U_i[u_{n+1}] = \gamma_i \qquad (i = 1, 2, \ldots;\ n = 0, 1, 2, \ldots) \tag{2.7}$$

angesetzt. Soweit die Funktionen $u_{n+1}(x)$ hierdurch bei stetigem $u_0(x)$ erklärt sind, genügen sie den Gleichungen (1.6) mit den hier definierten Funktionen $g(x)$ und $G(x, \xi)$.

Die Voraussetzungen über $g(x)$ können entsprechend wie in Nr. 1.1 gemildert werden: $g(x)$ braucht nicht stetig zu sein. Statt der Stetigkeit wird dann von der Lösung $u(x)$ einer Aufgabe der Form (2.3), (2.4) mit den inhomogenen Randbedingungen nur gefordert, daß $u(x)$ sich in der Gestalt (1.7) mit stetigem $\hat{u}(x)$ schreiben läßt, und die vorausgesetzten Differenzierbarkeitseigenschaften werden ebenfalls von $\hat{u}(x)$ verlangt. Auf diese Weise kann man weitere Randbedingungen erfassen. Sind z.B. auf Γ stückweise stetige Funktionswerte vorgeschrieben, so ist $g(x)$ auf $\overline{\mathfrak{B}}$ nicht stetig. Jedoch wollen wir diesen Fall der unstetigen Funktion $g(x)$ im folgenden nicht weiter berücksichtigen.

2.2. Ergebnis. Unter den in 2.1 genannten Voraussetzungen erhält man mit § 1 für die gegebene Aufgabe (2.1), (2.2) den

Satz 2. *a) Ist für eine im Gebiet $\mathfrak{G}$ verlaufende stetige Ausgangsnäherung $u_0(x)$ durch* (2.7) *eine erste Näherung $u_1(x)$ definiert, so gilt mit diesen Funktionen $u_0(x)$, $u_1(x)$ wörtlich der Satz 1*[7].

b) Im Falle

$$G(x, \xi) \geqq 0 \quad \textit{für} \quad x \neq \xi \qquad (x, \xi \in \overline{\mathfrak{B}}) \tag{2.8}$$

kann man statt (1.16) *fordern, daß $\sigma(x)$ eine Lösung der Beziehungen*

$$M[\sigma] \geqq \tilde{f}\big(x, \sigma(x) + \sigma_1(x) + \varrho_0(x)\big) - \tilde{f}\big(x, \varrho_0(x)\big), \tag{2.9}$$

$$U_i[\sigma] = 0 \qquad (i = 1, 2, \ldots) \tag{2.10}$$

mit stetigem $M[\sigma]$ sei.

Beweis. a) ist unmittelbar klar. Aus (2.9), (2.10) folgt mit (2.8)

$$\sigma = \int_{\mathfrak{B}} G\, M[\sigma]\, d\xi \geqq \int_{\mathfrak{B}} G\big[\tilde{f}(\xi, \sigma + \sigma_1 + \varrho_0) - \tilde{f}(\xi, \varrho_0)\big]\, d\xi,$$

so daß auch b) richtig ist.

Setzt man $w(x) = u_0(x)$ und $\varrho_0(x) \equiv 0$, so geht (1.16) in

$$\sigma(x) \geqq \int_{\mathfrak{B}} |G(x, \xi)|\, f\big(\xi, \sigma(\xi) + \sigma_1(\xi)\big)\, d\xi$$

und (2.9) entsprechend in

$$M[\sigma] \geqq f\big(x, \sigma(x) + \sigma_1(x)\big)$$

über. Bei linearer Majorante $\tilde{f}(x, y)$ erhält man immer diese einfacheren Ungleichungen.

Im allgemeinen ist es bei nichtnegativem $G(x, \xi)$ bequemer, $\sigma(x)$ als Lösung von (2.9), (2.10) zu bestimmen. Allerdings hat die Integralungleichung (1.16) mehr und darunter meistens auch einfachere Lösungen, denn es wird dabei nicht

[7] $g(x)$ und $G(x, \xi)$ bedeuten darin die in Nr. 2.1 erklärten Funktionen.

gefordert, daß $\sigma(x)$ den Randbedingungen (2.10) genügt. Diese Randbedingungen verhindern es z. B. im allgemeinen, daß eine Konstante als Lösung von (2.9), (2.10) zu verwenden ist. Siehe jedoch Zusatz 2.3.

Zusätze 2.1 und 2.2. *Die Zusätze 1.1 und 1.2 sind hier unmittelbar zu verwenden. Im Falle* (2.8) *einer nichtnegativen Funktion $G(x, \xi)$ kann man statt* (1.23) *auch die stärkere Forderung stellen, daß $W(x)$ eine Lösung der Beziehungen*

$$M[W] \geqq \frac{1}{\gamma} V(x), \qquad U_i[W] = 0 \qquad (i = 1, 2, \ldots)$$

mit stetigem $M[W]$ sei. Hat ferner die Aufgabe

$$M[\sigma] = \frac{\eta}{\gamma} V(x), \qquad U_i[\sigma] = 0 \qquad (i = 1, 2, \ldots)$$

im Falle (2.8) *eine Lösung $\sigma(x)$, so ist dies die in* (1.24) *stehende Schranke.*

Zusatz 2.3. *Die Bedingungen* (2.10) *lassen sich durch die Forderung ersetzen, daß die Aufgabe*

$$M[u] = 0, \qquad U_i[u] = U_i[\sigma] \qquad (i = 1, 2, \ldots) \tag{2.11}$$

eine nichtnegative Lösung besitze.

Diese Möglichkeit wird in den Nrn. 3.2 und 4.2 ausgenutzt.

Beweis. $\sigma(x)$ erfülle die Ungleichung (2.9), $M[\sigma]$ sei stetig, und die Aufgabe (2.11) habe eine nichtnegative Lösung $z(x)$. Für die Funktion $\hat{\sigma}(x) = \sigma(x) - z(x)$ gilt dann

$$M[\hat{\sigma}] = M[\sigma] \geqq r(x), \qquad U_i[\hat{\sigma}] = 0 \qquad (i = 1, 2, \ldots)$$

mit

$$r(x) = \tilde{f}(x, \sigma + \sigma_1 + \varrho_0) - \tilde{f}(x, \varrho_0).$$

Es ist daher

$$\hat{\sigma}(x) = \int\limits_{\mathfrak{B}} G(x, \xi)\, M[\sigma]\, d\xi \geqq \int\limits_{\mathfrak{B}} G(x, \xi)\, r(\xi)\, d\xi \geqq 0,$$

so daß man wegen $\hat{\sigma}(x) \leqq \sigma(x)$ weiter abschätzen kann:

$$r(x) \geqq \tilde{f}(x, \hat{\sigma} + \sigma_1 + \varrho_0) - \tilde{f}(x, \varrho_0).$$

$\hat{\sigma}(x)$ genügt also den in Satz 2 für $\sigma(x)$ geforderten Bedingungen (2.9), (2.10), und statt $\hat{\sigma}(x)$ ist auch $\sigma(x)$ als Fehlerschranke zu verwenden.

Bei manchen Aufgaben ist es auch nicht erforderlich, daß $u_1(x)$ die Randbedingungen genau erfüllt (s. z. B. Nr. 3.2).

2.3. Vergleich mit bekannten Abschätzungen. Erläuterung der Ergebnisse. Wir vergleichen die bewiesene Fehlerabschätzung (1.18) mit denjenigen, welche man mit Hilfe des bekannten Fixpunktsatzes für kontrahierende Operatoren [*12*], [*3*] bekommen kann. Zunächst nehmen wir an, es werde eine lineare Majorante verwendet. Dann liefert der Zusatz 1.2 die Schranken (1.26) und (1.27). Die gröbere dieser Schranken (1.27) kann man auch mit dem erwähnten Fixpunktsatz erhalten, indem man den numerischen Abstand

$$\delta(u, v) = \operatorname*{Max}_{\mathfrak{B}} \frac{|u(x) - v(x)|}{W(x)} \tag{2.12}$$

benutzt. Wählt man dabei wie hier als $\mathfrak{R}$ die Menge der stetigen Funktionen, so muß noch $W(x) > 0$ gefordert werden. (Es ist aber möglich, in (2.12) auch Funktionen $W(x)$ zuzulassen, die Nullstellen haben, indem man andere Funktionenmengen $\mathfrak{R}$ betrachtet. Derartige Schwierigkeiten treten hier jedoch nicht auf.) Die Schranke (1.26) stellt gegenüber (1.27) schon eine Verbesserung dar; z.B. hängt sie auch im Falle der Gewichtsfunktion $W(x) \equiv 1$ von x ab, während (1.27) eine Konstante liefert. Man erhält aber noch genauere Fehlerabschätzungen, wenn man $\sigma(x)$ direkt aus (1.16) bzw. (2.9), (2.10) berechnet.

Ist $f(x, y)$ linear, so wählt man auch eine lineare Funktion $\tilde{f}(x, y)$. Wir untersuchen nun, welche Vorteile entstehen können, wenn man bei nichtlinearem $f(x, y)$ eine nichtlineare Majorante $\tilde{f}(x, y)$ verwendet. Dazu besprechen wir zunächst die Wahl des Gebietes $\mathfrak{G}$ (1.3). Als $\mathfrak{G}$ kann man nicht immer dasjenige Gebiet ansetzen, in dem sich $f(x, y)$ überhaupt in vernünftiger Weise erklären läßt. Hängt die Majorante von $\mathfrak{G}$ ab [s. z.B. (1.11)], so ist es vielmehr zweckmäßig, ein „möglichst kleines" Gebiet $\mathfrak{G}$ zu benutzen[8]. Jedoch muß (1.17) [bzw. (1.20)] gelten. Praktisch geht man in diesem Falle so vor, daß man zunächst $u_0(x)$ und $u_1(x)$ ermittelt und auf Grund des Verlaufes dieser Funktionen $\mathfrak{G}$ so schätzt, daß es die Lösung $u^*(x)$ vermutlich enthalten wird. Genauer: man schätzt eine obere Schranke $S(x)$ für die sich mit Satz 1 oder Satz 2 ergebende Fehlerschranke $\sigma(x) \geqq |u^*(x) - u_1(x)|$ und richtet $\mathfrak{G}$, d.h. $\varphi(x)$ und $\psi(x)$, so ein, daß $\varphi(x) + S(x) \leqq u_1(x) \leqq \psi(x) - S(x)$ gilt, daß also $\mathfrak{G}$ auch den Streifen: $x \in \overline{\mathfrak{B}}$, $u_1(x) - S(x) \leqq y \leqq u_1(x) + S(x)$ enthält. (Auf die möglichen Variationen bei Benutzung des Zusatzes 1.1 gehen wir der Einfachheit halber nicht ein.) Hat man dann eine Fehlerschranke $\sigma(x)$ berechnet, sieht man oft, daß ein kleineres Gebiet $\mathfrak{G}$ und eine zugehörige günstigere Majorante die Voraussetzungen auch erfüllt hätten, und man kann, wenn man möglichst gute Fehlerschranken haben will, die Abschätzung mit der neuen Majorante wiederholen.

Verwendet man eine lineare Majorante $\tilde{f}(x, y)$, so muß man bei nichtlinearer Funktion $f(x, y)$ fast immer so vorgehen, denn eine solche Funktion f genügt im allgemeinen nicht im ganzen Zylinder

$$\mathfrak{Z}: \quad x \in \overline{\mathfrak{B}}, \qquad -\infty < y < \infty$$

einer Lipschitz-Bedingung (1.12) mit festem $a_1(x)$. Dies etwas umständliche Verfahren entfällt, wenn man eine nichtlineare Funktion $\tilde{f}(x, y)$ wählt, mit der (1.5) in ganz $\mathfrak{Z}$ gilt, was sehr oft möglich ist. Dann kann man $\mathfrak{G} = \mathfrak{Z}$ setzen und braucht (1.17) nicht nachzuprüfen. In diesem Sinne ist also die Rechnung bei nichtlinearer Funktion $\tilde{f}(x, y)$ einfacher, falls $f(x, y)$ nichtlinear ist (vgl. auch Beispiel 2).

Allerdings sind die zu lösenden Ungleichungen (1.16) bzw. (2.9) dann in $\sigma(x)$ nichtlinear. Man kann deren Lösungen nicht überlagern. Bei den in den §§ 3 und 4 gerechneten Beispielen mit nichtlinearer Majorante gelang es ohne Schwierigkeiten, brauchbare Schranken zu ermitteln. In anderen Fällen kann die Nichtlinearität u.U. stören. Dann wird folgendes Verfahren vorgeschlagen.

---

[8] Dagegen wählt man $\vartheta(x)$ in (1.4) immer so groß wie möglich, im allgemeinen kann man $\vartheta = \infty$ setzen.

Man schätzt eine Schranke $S(x)$ für die Lösung $\sigma(x)$ der Aufgabe (1.16) bzw. (2.9), (2.10) — das entspricht also dem oben beschriebenen Vorgang — und ersetzt $\sigma(x)$ auf der rechten Seite von (1.16) bzw. (2.9) teilweise durch $S(x)$, so daß eine in $\sigma(x)$ lineare (inhomogene) Funktion bleibt, z.B. $(\sigma+\sigma_1)^3 \leqq (\sigma+\sigma_1)(S+\sigma_1)^2$. Besitzt die so linearisierte Ungleichung eine Lösung $\sigma(x) \leqq S(x)$, so löst diese Funktion auch die ursprüngliche Ungleichung, und man kann das Verfahren evtl. mit der berechneten Funktion $\sigma(x)$ als neuer Schranke $S(x)$ wiederholen. Bei diesem Verfahren braucht man nicht an allen Stellen die gleiche Schranke $S(x)$ zu verwenden und man nutzt auf diese Weise unter anderem den Vorteil nichtlinearer Funktionen $\tilde{f}(x, y)$ aus, oft bessere Fehlerabschätzungen zu liefern.

Wir erläutern, wann mit nichtlinearen Majoranten genauere Fehlerabschätzungen zu erwarten sind. Der Einfachheit halber betrachten wir nur Funktionen $\tilde{f}(x, y)$, welche mit Hilfssatz 1 ermittelt werden können. Ist $\tilde{f}(x, y)$ linear, so hängt $\tilde{f}_y(x, |y|)$ in (1.8) nicht von y ab, eine nichtlineare Funktion $\tilde{f}(x, y)$ dagegen kann das Anwachsen der Funktion $|f_y(x, y+w)|$ mit y oder $-y$ berücksichtigen. Je stärker $|f_y(x, y+w)|$ mit y oder $-y$ zunimmt, desto günstiger ist es im allgemeinen, eine nichtlineare Majorante zu verwenden. — Erfüllt $f(x, y)$ die Voraussetzungen des Hilfssatzes 2, ist $G(x, \xi) \geqq 0$ und $w \leqq u_0 \leqq u_1$ und verwendet man $\sigma_1 = u_1 - u_0$, $\varrho_0 = u_0 - w$ und die Majorante (1.10), so ist $\sigma = u^* - u_1$[9] Lösung der Aufgabe (2.9), (2.10) mit dem Gleichheitszeichen in (2.9). In diesem Falle sind also die hier abgeleiteten Fehlerabschätzungen für $|u^* - u_1|$ nicht weiter zu verschärfen, und durch genügend sorgfältige Berechnung einer Schranke $\sigma(x)$ aus (2.9), (2.10) kann man — jedenfalls theoretisch — beliebig nahe an $|u^* - u_1|$ herankommen. Ein solches Resultat ist bei nichtlinearem $f(x, y)$ offensichtlich nur möglich, wenn man auch nichtlineare Majoranten $\tilde{f}(x, y)$ betrachtet. — Natürlich hängt es stark von den speziellen Eigenschaften des gerade vorliegenden Problems ab, welche Majorante zu wählen ist.

2.4. Das vereinfachte Newtonsche Verfahren. Es sei nun eine Differentialgleichungsaufgabe

$$L[u] = h(x, u), \qquad U_i[u] = \gamma_i \qquad (i = 1, 2, \ldots) \tag{2.13}$$

mit linearem, homogenem Differentialoperator L gegeben. Man kann versuchen, sie mit dem Iterationsverfahren

$$L[u_{n+1}] = h(x, u_n), \qquad U_i[u_{n+1}] = \gamma_i \qquad (i = 1, 2, \ldots;\ n = 0, 1, 2, \ldots) \tag{2.14}$$

zu lösen und unsere Ergebnisse für $M = L$ und $f = h$ anzuwenden. Oft ist es jedoch zweckmäßig, die gegebene Aufgabe umzuformen, M und f also anders zu definieren. Will man dann die obigen Ergebnisse benutzen, muß man nachprüfen, ob die geforderten Voraussetzungen mit den verwendeten Größen M und f erfüllt sind.

Das „an der Stelle $w(x)$ angesetzte vereinfachte Newtonsche Verfahren“ (s. [2] bei $w = u_0$) für die Aufgabe (2.13) hat die Gestalt (2.7) mit

$$M[u] = L[u] - h_y\big(x, w(x)\big)\, u, \qquad f(x, y) = h(x, y) - h_y\big(x, w(x)\big)\, y. \tag{2.15}$$

[9] Aus den in [4] genannten Ergebnissen folgt hier $u_1(x) \leqq u^*(x)$.

Es ist im allgemeinen schwieriger durchzuführen als das Verfahren (2.14), liefert aber vielfach bessere Werte und führt bei geeigneter Funktion $w(x)$ oft auch dann zum Ziel, wenn (2.14) nicht konvergiert.

Das vereinfachte Newtonsche Verfahren läßt sich als Verfahren der Form (2.7) mit $f_y(x, w(x)) = 0$ charakterisieren. Wegen dieses Verschwindens der ersten Ableitung ist es besonders zweckmäßig, eine nichtlineare Majorante zu verwenden. Sind $h(x, y)$ und $h_y(x, y)$ auf $\mathfrak{G}$ (1.3) stetig und gilt

$$\begin{gathered}(|f_y(x,y) - f_y(x,z)| =) \quad |h_y(x,y) - h_y(x,z)| \leqq a_2(x)\,|y-z| \\ \text{für} \quad x \in \overline{\mathfrak{B}}, \quad \varphi(x) \leqq y,\ z \leqq \psi(x)\end{gathered} \tag{2.16}$$

mit einer stetigen Funktion $a_2(x)$, so erfüllt nach Hilfssatz 3 z.B.

$$\tilde{f}(x,y) = \tfrac{1}{2}\,a_2(x)\,y^2 \tag{2.17}$$

die Voraussetzungen (1.5).

Im allgemeinen benutzt man $w = u_0$. Durch geeignete Wahl einer anderen, u_0 benachbarten Funktion $w(x)$ kann man jedoch manchmal erreichen, daß der Ausdruck $h_y(x, w(x))$ und damit die Iterationsvorschrift einfacher wird.

Eine weitere Möglichkeit, eine einfachere Iterationsvorschrift zu bekommen, die Vorteile des Newtonschen Verfahrens jedoch im wesentlichen beizubehalten, besteht darin, $h_y(x, w(x))$ durch eine stetige Näherungsfunktion $q(x)$ zu ersetzen, u.U. eine Konstante. Das entsprechende Iterationsverfahren ist dann (2.7) mit

$$M[u] = L[u] - q(x)\,u, \qquad f(x,y) = h(x,y) - q(x)\,y.$$

Statt (2.17) erhält man unter der Voraussetzung (2.16) die Majorante

$$\tilde{f}(x,y) = a_1(x)\,y + \tfrac{1}{2}\,a_2(x)\,y^2,$$

wobei $a_1(x)$ eine stetige Funktion mit

$$|h_y(x, w(x)) - q(x)| \leqq a_1(x)$$

ist. Das hier auftretende lineare Glied hat um so weniger Gewicht, je besser die Näherung $q(x)$ ist.

Beim vereinfachten Newtonschen Verfahren wird man oft $u_0(x)$ rückwärts von $u_1(x)$ ausgehend berechnen. Besonders zweckmäßig erscheint dies z.B. im Spezialfall

$$h(x,y) = p_2(x)\,y^2 + p_1(x)\,y + p_0(x) \tag{2.18}$$

mit stetigen Funktionen $p_i(x)$ und $p_2(x) \neq 0$. Hier wird für $w = u_0$

$$\begin{aligned} M[u] &= L[u] - \big(2p_2(x)\,u_0(x) + p_1(x)\big)\,u, \\ f(x,y) &= p_2(x)\,y^2 - 2p_2(x)\,u_0(x)\,y + p_0(x). \end{aligned} \tag{2.19}$$

Mit (2.15) und (2.18) geht die Differentialgleichung für $u_1(x)$ über in

$$L[u_1] - (2p_2 u_0 + p_1)\,u_1 = -p_2 u_0^2 + p_0.$$

Eine Lösung u_0 bei gegebenem u_1 ist

$$u_0(x) = u_1(x) - \operatorname{sgn} p_2(x) \cdot \sqrt{\frac{\varepsilon(x)}{p_2(x)}},$$

wobei $\varepsilon(x)$ den Defekt der Differentialgleichung in (2.13)

$$\varepsilon(x) = -L[u_1] + p_2 u_1^2 + p_1 u_1 + p_0$$

bedeutet.

Als Majorante (bezüglich $w = u_0$) läßt sich nach (2.16), (2.17) die für $x \in \overline{\mathfrak{B}}$, $0 \leqq y < \infty$ erklärte Funktion

$$\tilde{f}(x, y) = |p_2(x)|\, y^2$$

verwenden. Mit $\sigma_1(x) = \operatorname{sgn} p_2(x) \cdot (u_1(x) - u_0(x))$ und $\varrho_0(x) = 0$ lauten die Beziehungen (2.9), (2.10)

$$L[\sigma] - (2p_2 u_1 + p_1)\,\sigma - |p_2|\,\sigma^2 \geqq |\varepsilon(x)|, \qquad U_i[\sigma] = 0 \qquad (i = 1, 2, \ldots). \qquad (2.20)$$

Im Spezialfall (2.18) bestimme man $u_1(x)$ also so, daß diese Funktion die Randbedingungen $U_i[u_1] = \gamma_i$ erfüllt und $\varepsilon(x) \cdot \operatorname{sgn} p_2(x)$ nichtnegativ und möglichst klein ausfällt. Als Fehlerabschätzung erhält man dann $|u^*(x) - u_1(x)| \leqq \sigma(x)$ mit einer (2.20) genügenden Funktion $\sigma(x)$, wenn für M und f in (2.19) die allgemein geforderten Voraussetzungen gelten und die zugehörige Funktion $G(x, \xi)$ nichtnegativ ist. Im Falle $p_2(x) > 0$ erfüllt $\sigma(x) = u^*(x) - u_1(x)$ die Bedingungen (2.20) mit dem Gleichheitszeichen in der Differentialungleichung.

§ 3. Beispiele für Randwertaufgaben

3.1. Beispiel einer gewöhnlichen Differentialgleichung. Man kann leicht gewisse Klassen von Randwertaufgaben bei gewöhnlichen Differentialgleichungen angeben, für welche die in Nr. 2.1 genannten Voraussetzungen erfüllt sind, insbesondere also eine Greensche Funktion $G(x, \xi)$ mit den geforderten Eigenschaften existiert (zur Greenschen Funktion s. etwa [*1*], [*6*]; (1.1) ist z.B. erfüllt, falls $G(x, \xi)$ auf $\overline{\mathfrak{B}} \times \overline{\mathfrak{B}}$ stetig ist). Wir beschränken uns hier darauf, ein numerisches Beispiel zu rechnen.

Beispiel 1:

$$(M[u] =)\; -u'' = 3\,x\,u^2 - u^3\; (= f(x, u)), \qquad u(0) = \tfrac{1}{2}, \qquad u(1) = 1\,.$$

Das übliche Mehrstellen-Differenzenverfahren (s. [*3*] oder [*9*])

$$[1 \;\; -2 \;\; 1]_i\, u + \frac{h^2}{12}\,[1 \;\; 10 \;\; 1]_i\, f(x, u) = 0$$

zur Schrittweite $h = \frac{1}{8}$ ergab die in Tabelle 1, Spalte 2 stehenden Näherungswerte $\tilde{u}(x_i)$ $(x_i = i\,h)$. Das bei diesem Verfahren auftretende nichtlineare Gleichungssystem wurde iterativ nach einer in [*9*] beschriebenen Methode gelöst. Die dabei erforderliche Rechenarbeit war sehr gering, da die benutzte Methode für die Rechenanlage IBM 650 programmiert ist. Auch für die weiteren Rechnungen wurde die Rechenanlage herangezogen.

Mit Hilfe der Ausgleichsrechnung wurden dann Konstanten a, b so bestimmt, daß die Werte $u_0(x_i)$ der Funktion

$$u_0(x) = \tfrac{1}{2}(1 + x) + x(1 - x)(a + b\,x)$$

die $\tilde{u}(x_i)$ im Sinne der Fehlerquadratmethode möglichst gut annähern (s. Tabelle 1, Spalte 3):

$$a = 0{,}097858, \qquad b = 0{,}372027\,.$$

Von dieser Funktion $u_0(x)$ ausgehend erhält man nach der Iterationsvorschrift

$$-u''_{n+1} = 3\,x\,u_n^2 - u_n^3, \qquad u_{n+1}(0) = \tfrac{1}{2}, \qquad u_{n+1}(1) = 1 \qquad (u = 0, 1, 2, \ldots)$$

die erste Näherung

$$u_1(x) = \tfrac{1}{2}\,(1 + x) + x(1 - x) \sum_{j=0}^{9} \alpha_j\, x^j$$

(Tabelle 1, Spalte 4 und Tabelle 2). Es ist

$$|u_1(x) - u_0(x)| \leqq \sigma_1(x) \quad \text{mit} \quad \sigma_1(x) = x(1 - x)\,\alpha, \qquad \alpha = 0{,}0367.$$

Tabelle 1 (Beispiel 1)

i	$\tilde{u}(x_i)$	$u_0(x_i)$	$u_1(x_i)$	$\sigma(x_i)$
0	0,5	0,5	0,5	0,0
1	0,580 145	0,578 290	0,580 166	0,001 590
2	0,661 302	0,660 787	0,661 328	0,002 923
3	0,741 769	0,743 133	0,741 798	0,003 804
4	0,818 845	0,820 968	0,818 874	0,004 111
5	0,888 686	0,889 932	0,888 699	0,003 804
6	0,946 269	0,945 665	0,946 251	0,002 923
7	0,985 564	0,983 807	0,985 529	0,001 590
8	1,0	1,0	1,0	0,0

Tabelle 2 (Beispiel 1)

j	α_j
0	0,134 487
1	0,196 987
2	0,146 720
3	0,059 070
4	— 0,014 350
5	— 0,018 614
6	— 0,005 782
7	— 0,002 688
8	— 0,000 797
9	0,000 468

Zur Fehlerabschätzung ermitteln wir eine einfache lineare Majorante. Im Gebiet

$$\mathfrak{G}: \quad 0 \leqq x \leqq 1, \qquad x \leqq y \leqq 1,$$

welches $u_0(x)$ und $u_1(x)$ enthält, gilt

$$|f_y(x, y)| = 3\,|y|\,|2x - y| \leqq 3,$$

so daß wir hierfür nach Hilfssatz 3 die Funktion

$$\tilde{f}(x, y) = 3\,y$$

benutzen können. Alle erforderlichen Voraussetzungen sind erfüllt, insbesondere gilt auch (2.8). Man bekommt für $\sigma(x)$ die linearen Bedingungen

$$-\sigma'' \geqq 3\sigma + 3\sigma_1, \tag{3.1}$$

$$\sigma(0) = \sigma(1) = 0. \tag{3.2}$$

Die Funktion

$$\sigma(x) = \frac{2}{3}\,\frac{\alpha}{\sin\sqrt{3}} \left\{\sin\sqrt{3}\,x + \sin\sqrt{3}\,(1 - x) - \sin\sqrt{3}\right\} - \alpha\,x(1 - x)$$

(Tabelle 1, Spalte 5)

erfüllt diese Forderungen, und zwar mit dem Gleichheitszeichen in (3.1). Es gilt auch (1.17)

$$x + \sigma(x) \leqq u_1(x) \leqq 1 - \sigma(x). \tag{3.3}$$

Nach Satz 2 existiert also eine Lösung $u^*(x)$ mit

$$|u^*(x) - u_1(x)| \leqq \sigma(x) \qquad (\leqq 0{,}00412).$$

Ein Ansatz

$$\sigma(x) = A\,x(1-x)\,[1 + x(1-x)] \qquad (A = \text{const})$$

zur Lösung der Aufgabe (3.1), (3.2), der so gewählt ist, daß σ'' den Faktor $x(1-x)$ enthält, führt auf $A = \frac{4}{11}\alpha$. Auch für diese Schranke $\sigma(x)$ gilt (3.3), und es ist hierfür $\sigma(x) \leqq 0{,}00417$.

Eine nichtlineare Majorante zu verwenden, hat bei dieser Aufgabe nicht viel Sinn, weil $|f_y(x, y)|$ das in der obigen linearen Funktion $\tilde{f}(x, y)$ benutzte Maximum $3 = \underset{\mathfrak{G}}{\mathrm{Max}}|f_y(x, y)|$ für $x = 1$ und den Wert $y = u^*(1) = 1$ der Lösungsfunktion annimmt. Eine optimale nichtlineare Majorante $\tilde{f}(x, y)$ ist auch von komplizierterer Gestalt, da $f_y(x, y)$ in jedem die Lösung enthaltenden Gebiet das Vorzeichen wechselt. Der günstigste Wert A, welchen man mit einer optimalen Funktion $\tilde{f}(x, y)$ und dem zuletzt genannten Ansatz für $\sigma(x)$ bekommen kann, ist $A = \frac{1}{3}\alpha$, mit dem dann $\sigma(x) \leqq 0{,}00383$ ist.

3.2. Eine Klasse partieller Differentialgleichungen. Wir nennen nun eine Klasse von Randwertaufgaben bei partiellen Differentialgleichungen, für welche die geforderten Voraussetzungen gelten.

Es sei $\mathfrak{B}$ ein offenes, beschränktes, einfach zusammenhängendes Gebiet der x_1, x_2-Ebene (kurz: x-Ebene) mit stetig gekrümmtem Rand Γ: $x_1 = \alpha(s)$, $x_2 = \beta(s)$ (s = Bogenlänge; $\alpha''(s)$, $\beta''(s)$ stetig [7]). Gesucht ist eine auf $\overline{\mathfrak{B}} = \mathfrak{B} - \Gamma$ stetige, auf $\mathfrak{B}$ mit stetigen ersten und zweiten partiellen Ableitungen versehene Funktion $u(x) = u(x_1, x_2)$, welche auf $\mathfrak{B}$ die Differentialgleichung

$$\big(M[u] =\big) -\Delta u = f(x, u) \tag{3.4}$$

erfüllt und auf Γ gegebene Werte

$$u = \gamma(s) \tag{3.5}$$

annimmt, wobei $\gamma(s)$ eine stetig differenzierbare Funktion bedeute und $f(x, y) = f(x_1, x_2; y)$ die in Nr. 1.1 unter c) genannten Eigenschaften besitze. Die Iterationsvorschrift lautet

$$-\Delta u_{n+1} = f(x, u_n) \quad \text{auf } \mathfrak{B}, \qquad u_{n+1} = \gamma(s) \quad \text{auf } \Gamma \qquad (n = 0, 1, 2, \ldots). \tag{3.6}$$

Aus den in [5] und [7] dargestellten Ergebnissen kann man schließen, daß eine stetige Funktion $g(x)$ und eine nichtnegative Greensche Funktion $G(x, \xi)$ existieren, welche die in Nr. 2.1 genannten Voraussetzungen erfüllen. Ausführlicher ist dies in [4] dargestellt, wo Probleme der Art (3.4), (3.5) in anderer Weise behandelt werden.

Die Vergleichsaufgabe lautet hier

$$-\Delta\sigma \geqq \tilde{f}(x, \sigma + \sigma_1 + \varrho_0) - \tilde{f}(x, \varrho_0) \quad \text{auf } \mathfrak{B}, \qquad \sigma \geqq 0 \quad \text{auf } \Gamma.$$

Man braucht nach Zusatz 2.3 auf Γ nur $\sigma \geqq 0$ statt $\sigma = 0$ zu fordern, da die Lösung z der Aufgabe

$$-\Delta u = 0 \quad \text{auf } \mathfrak{B}, \qquad u = \sigma \quad \text{auf } \Gamma$$

für $\sigma \geqq 0$ nach dem hier gültigen Randmaximumsatz (s. [3]) nichtnegativ ist.

Die Funktion $u_1(x)$ ist Lösung einer Aufgabe

$$-\Delta u = r(x) \quad \text{auf } \mathfrak{B}, \quad u = \gamma(s) \quad \text{auf } \Gamma$$

mit bekannter stetiger Funktion $r(x)$. Kennt man eine Funktion $\hat{u}_1(x)$ welche die für $u_1(x)$ geforderte Differentialgleichung erfüllt, nicht jedoch die Randbedingung, so kann man auch den Fehler $|u^*(x) - \hat{u}_1(x)|$ abschätzen, denn nach dem Randmaximumsatz gilt

$$|u_1(x) - \hat{u}_1(x)| \leqq \omega \quad \text{mit} \quad \omega = \operatorname*{Max}_{\Gamma} |\hat{u}_1 - \gamma(s)|,$$

also

$$|u_1(x) - u_0(x)| \leqq \sigma_1(x) \quad \text{mit} \quad |\hat{u}_1(x) - u_0(x)| + \omega \leqq \sigma_1(x) \tag{3.7}$$

und

$$|u^*(x) - \hat{u}_1(x)| \leqq |u^*(x) - u_1(x)| + \omega \leqq \sigma(x) + \omega,$$

wobei $\sigma(x)$ mit der Schranke $\sigma_1(x)$ aus (3.7) zu berechnen ist.

3.3. Beispiel einer partiellen Differentialgleichung.

Beispiel 2 ist eine Aufgabe der in Nr. 3.2 behandelten Art:

$$-\Delta u = \alpha + x_1^2 u^2 \quad \text{für} \quad 0 \leqq r < 1,$$
$$u = 0 \qquad \text{für} \quad r = 1 \quad \left(r = \sqrt{x_1^2 + x_2^2}\right)$$

mit einer positiven Konstanten α.

Von

$$u_0 = a(1 - r^2)$$

ausgehend berechnet man — $x_1 = r\cos\varphi$ setzend —

$$u_1 = (1 - r^2)\left\{\frac{\alpha}{4} + \frac{a^2}{1152}(13 + 13r^2 - 23r^4 + 9r^6) + \frac{a^2\cos 2\varphi}{480}(9r^2 - 11r^4 + 4r^6)\right\}.$$

Fordert man $u_1(0,0) = u_0(0,0)$, so ergibt sich (s. Tabelle 3, Spalte 2)

$$a = a(\alpha) = \frac{576}{13}\left(1 - \sqrt{1 - \frac{13\alpha}{1152}}\right) \quad \text{für} \quad \alpha \leqq \frac{1152}{13}$$

und

$$|u_1 - u_0| \leqq (1 - r^2)\, a^2 \left\{\frac{1}{1152}|13r^2 - 23r^4 + 9r^6| + \frac{1}{480}|9r^2 - 11r^4 + 4r^6|\right\} \leqq \sigma_1$$

mit

$$\sigma_1 = r^2(1 - r^2)\, a^2 d \quad \text{und} \quad d = \frac{13}{1152} + \frac{9}{480} = \frac{281}{5760}.$$

Bei der Fehlerabschätzung verwenden wir nun verschiedenartige Majoranten.

a) Im Gebiet

$$\mathfrak{G}\colon \quad 0 \leqq r \leqq 1, \quad -\infty < y < \infty$$

gilt für $f(x, y) = \alpha + x_1^2 y^2$:

$$|f_y(x, u_0 + y)| = 2x_1^2|u_0 + y| \leqq 2r^2(u_0 + |y|) = \tilde{f}_y(x, |y|)$$

mit

$$\tilde{f}(x, y) = \tilde{f}(x_1, x_2; y) = 2r^2 u_0 y + r^2 y^2,$$

welche Funktion man daher nach Hilfssatz 1 für $w = u_0$ als Majorante verwenden kann. Die Vergleichsaufgabe lautet damit bei $\varrho_0 = 0$:

$$
\begin{aligned}
&-\Delta\sigma - \left[2a\,r^2(1-r^2) + 2a^2 d\,r^4(1-r^2)\right]\sigma - r^2\sigma^2 \\
&\qquad \geqq 2a^3 d\,r^4(1-r^2)^2 + a^4 d^2 r^6 (1-r^2)^2 \quad \text{für} \quad 0 \leqq r < 1, \\
&\sigma = 0 \quad \text{für} \quad r = 1.
\end{aligned}
$$

Der Ansatz $\sigma = A(1-r^4)$ $(A = \text{const})$ führt auf die Ungleichung

$$
\begin{aligned}
&A\left[16 - 2a(1-r^2)(1-r^4) - 2a^2 d\,r^2(1-r^2)(1-r^4)\right] - A^2(1-r^4)^2 \\
&\qquad \geqq 2a^3 d\,r^2(1-r^2)^2 + a^4 d^2 r^4 (1-r^2)^2 \quad \text{für} \quad 0 \leqq r < 1.
\end{aligned}
$$

Mit

$$
\begin{aligned}
&(1-r^2)(1-r^4) \leqq 1, \qquad r^2(1-r^2)(1-r^4) \leqq \tfrac{1}{4}\,^{10}, \\
&(1-r^4)^2 \leqq 1, \qquad r^2(1-r^2)^2 \leqq \tfrac{4}{27}, \qquad r^4(1-r^2)^2 \leqq \tfrac{1}{16}
\end{aligned}
$$

erhält man die quadratische Gleichung

$$
A^2 - A\left[16 - a\left(2 + \tfrac{1}{2}a\,d\right)\right] + a^2\left(\tfrac{8}{27}a\,d + \tfrac{1}{16}a^2 d^2\right) = 0
$$

als hinreichende Bedingung für A. Eine nichtnegative Lösung $A(\alpha)$ existiert für $\alpha \leqq \alpha_0 \approx 21{,}8$ (Tabelle 3, Spalte 3). Nach Satz 2 besitzt also für solche Werte α auch die gegebene Aufgabe eine Lösung u^* und es gilt die Fehlerabschätzung

$$
\left|u^*(x_1, x_2) - u_1(x_1, x_2)\right| \leqq A(\alpha)\,(1-r^4) \quad \text{für} \quad \alpha \leqq \alpha_0 \approx 21{,}8.
$$

Tabelle 3 (Beispiel 2)

α	$u_1(0,0) = a(\alpha)$	$(\lvert u^*(0,0) - u_1(0,0)\rvert \leqq)$ $A(\alpha)$	$(\lvert u^*(0,0) - u_1(0,0)\rvert \leqq)$ $B(\alpha)$
2	0,502852	0,000123	0,000126
4	1,01155	0,00108	0,00113
6	1,52629	0,00405	0,00437
8	2,0473	0,0107	0,0120
10	2,5748	0,0237	0,0281
12	3,1091	0,0471	0,0601
14	3,650	0,088	0,127
16	4,199	0,159	0,313
18	4,755	0,289	—
20	5,319	0,578	—
21,8	5,83	1,60	—

b) Wir benutzen nun eine lineare Majorante und schränken $\mathfrak{G}$ ein auf

$$
\mathfrak{G}: \quad 0 \leqq r \leqq 1, \qquad 0 \leqq y \leqq c(1-r^2)
$$

mit noch zu wählender Konstanten c. In diesem Gebiet $\mathfrak{G}$ ist

$$
\left|f_y(x, y)\right| \leqq 2c\,r^2(1-r^2),
$$

[10] Schätzt man hier genauer ab, bekommt man nur wenig bessere Ergebnisse.

so daß man

$$\tilde{f}(x, y) = 2c\,r^2(1-r^2)\,y \tag{3.8}$$

und das Vergleichsproblem

$$-\Delta\sigma - 2c\,r^2(1-r^2)\,\sigma \geqq 2a^2 c\,d\,r^4(1-r^2)^2 \quad \text{für} \quad 0 \leqq r < 1,$$

$$\sigma(0) = 0 \qquad\qquad \text{für} \quad r = 1$$

verwenden kann. Der Ansatz $\sigma = B(1-r^4)$ ($B = \text{const}$) führt auf

$$B[16 - 2c(1-r^2)(1-r^4)] \geqq 2a^2 c\,d\,r^2(1-r^2)^2.$$

Wegen

$$(1-r^2)(1-r^4) \leqq 1, \qquad r^2(1-r^2)^2 \leqq \tfrac{4}{27} \tag{3.9}$$

genügt

$$B = \frac{2a^2 c\,d \cdot \frac{4}{27}}{16-2c}$$

für $c < 8$ dieser Ungleichung.

c muß so bestimmt werden, daß die Bedingung (1.17) erfüllt ist. Dies führt auf die Forderung

$$B(1-r^4) \leqq u_1(x) \leqq c(1-r^2) - B(1-r^4). \tag{3.10}$$

Wir schätzen u_1 in entsprechender Weise ab wie die unter a) aufgetretenen Funktionen:

$$u_1 \leqq \sigma_1 + u_0 \leqq a^2 d\,r^2(1-r^2) + a(1-r^2)$$

und bestimmen c aus der für die zweite der Ungleichungen (3.10) hinreichenden Bedingung

$$a^2 d\,r^2(1-r^2) + a(1-r^2) \leqq c(1-r^2) - B(1-r^4)$$

zu

$$c = a^2 d + a + 2B.$$

Setzt man diesen Wert in (3.9) ein, so ergibt sich für B die quadratische Gleichung

$$B^2 - B\left[4 - a\left(\tfrac{1}{2} + \tfrac{35}{54}a\,d\right)\right] + a^2\left(\tfrac{2}{27}a\,d + \tfrac{2}{27}a^2 d^2\right) = 0,$$

welche für

$$\alpha \leqq \alpha_1 \approx 16{,}6$$

eine nichtnegative Lösung besitzt.

Bei unmittelbarer Anwendung des Satzes 2 muß man noch die erste der Ungleichungen (3.10) nachprüfen. Darauf kann man nach Zusatz 1.1 jedoch verzichten. Wegen $g(x) \equiv 0$, $G(x, \xi) \geqq 0$ und $f(x, y) \geqq 0$ gilt die erste der Ungleichungen (1.21) nämlich mit $\Phi(x) \equiv 0$, und für diese Funktion $\Phi(x)$ ist

$$\operatorname{Max}[\Phi(x), u_1(x) - \sigma(x)] \geqq 0 = \varphi(x).$$

Man erhält damit die Fehlerabschätzung (s. Tabelle 3, Spalte 4)

$$|u^*(x_1, x_2) - u_1(x_1, x_2)| \leqq B(\alpha)(1-r^4) \quad \text{für} \quad \alpha \leqq \alpha_1 \approx 16{,}6.$$

Bei dieser Aufgabe konnten wir durch Mitführen eines Parameters c ein nahezu optimales Gebiet $\mathfrak{G}$ der angesetzten Form bekommen, brauchten also nicht, wie in Nr. 2.3 beschrieben, zu probieren. Trotzdem erscheint der Rechengang unter a) einfacher.

§ 4. Beispiele für Anfangswertaufgaben

4.1. Gewöhnliche Differentialgleichungen. Wir behandeln Aufgaben der Gestalt

$$\begin{aligned}(M[u]=)\, u' - p(x)\, u &= f(x, u)\\ u(0) &= a\end{aligned} \tag{4.1}$$

und fragen nach (stetig differenzierbaren) Lösungen im Intervall $\mathfrak{B}=[0,\alpha]$. $p(x)$ sei stetig und $f(x, y)$ erfülle die in Nr. 1.1 unter c) genannten Voraussetzungen.

Der gegebenen Aufgabe äquivalent ist die Integralgleichung (2.6) mit den Funktionen

$$g(x) = a\, e^{\int\limits_0^x p(\xi)\, d\xi} \quad \text{und} \quad G(x,\xi) = \begin{cases} e^{\int\limits_\xi^x p(\eta)\, d\eta} & \text{für} \quad \xi \leqq x, \\ 0 & \text{für} \quad \xi > x, \end{cases}$$

welche die in Nr. 2.1 verlangten Eigenschaften besitzen. Die Vergleichsaufgabe lautet

$$\begin{aligned}&\sigma' - p(x)\,\sigma \geqq \tilde{f}(x, \sigma + \sigma_1 + \varrho_0) - \tilde{f}(x, \varrho_0) \quad \text{für} \quad 0 \leqq x \leqq \alpha,\\ &\sigma(0) = 0.\end{aligned}$$

Wir rechnen ein einfaches Beispiel.

Beispiel 3:

$$u' = x^2 + u^2 \quad \text{für} \quad 0 \leqq x \leqq \alpha, \qquad u(0) = 1. \tag{4.2}$$

Die Aufgabe soll mit dem bei $w = u_0$ angesetzten vereinfachten Newtonschen Verfahren gelöst werden. Diesem Verfahren entspricht die Umformung der Differentialgleichung in die Gestalt

$$(M[u]=)\, u' - 2u_0 u = x^2 + [(u-u_0)^2 - u_0^2] \qquad (= f(x,u)).$$

Die Funktion $h(x, u)$ auf der rechten Seite der Differentialgleichung in (4.2) ist von der speziellen Form (2.18). Wir benutzen das für diesen Fall am Schluß der Nr. 2.4 beschriebene Rechenverfahren, haben also $u_1(x)$ so zu bestimmen, daß

$$\varepsilon(x) = -u_1' + x^2 + u_1^2$$

im Intervall $[0, \alpha]$ nichtnegativ und möglichst klein ausfällt, und erhalten dann die Fehlerschranke $\sigma(x)$ aus (2.20):

$$\sigma' - 2u_1(x)\,\sigma - \sigma^2 \geqq \varepsilon(x) \quad \text{für} \quad 0 \leqq x \leqq \alpha, \qquad \sigma(0) = 0. \tag{4.3}$$

Als sehr einfache Näherungsfunktion $u_1(x)$ verwenden wir zur Erläuterung der Methode zunächst den Abschnitt

$$u_1 = 1 + x + x^2 \qquad \text{(Tabelle 4, Spalte 2)}$$

der Potenzreihenentwicklung der exakten Lösung $u^*(x)$. Dann ergibt sich

$$\varepsilon(x) = 4x^2 + 2x^3 + x^4,$$

und der Ansatz $\sigma = C x^3$ ($C = \text{const}$) zur Lösung von (4.3) führt auf die Forderung

$$Q(x, C) = C^2 x^4 - C\left[3 - 2(x + x^2 + x^3)\right] + 4 + 2x + x^2 \leqq 0 \quad \text{für} \quad 0 \leqq x \leqq \alpha,$$

d.h. $Q(\alpha, C) \leqq 0$. Wir setzen $Q(\alpha, C) = 0$. Für $\alpha \leqq \alpha_2 \approx 0{,}5095$ existiert eine nichtnegative Lösung $C(\alpha)$ dieser Gleichung.

Für jedes solche α und $x \leqq \alpha$ erhält man also die Fehlerschranke $\sigma = x^3 \cdot C(\alpha)$. Indem man $x = \alpha$ verwendet, ergibt sich die Abschätzung

$$|u^*(x) - u_1(x)| \leqq x^3 C(x) \quad \text{für} \quad 0 \leqq x \leqq \alpha_2 \approx 0{,}5095 \qquad \text{(Tabelle 4, Spalte 3)}.$$

Die Potenzreihenentwicklung der Schranke $x^3 C(x)$ nach x beginnt mit $\frac{4}{3}x^3$, dem nächsten Glied in der Taylor-Reihe der Lösung. Theoretisch kann man hier — wie auch bei der Funktion (4.4) — eine beliebig genaue Schranke erhalten, denn für $\sigma = u^* - u_1$ — es ist $u_1 \leqq u^*$ — gilt in der Differentialungleichung von (4.3) das Gleichheitszeichen.

Tabelle 4 (Beispiel 3)

x	$u_1 = 1 + x + x^2$	$(\|u^*(x) - u_1(x)\| \leqq)\ x^3 C(x)$	$\hat{u}_1 = \frac{1}{1-x} + \frac{x^3}{3}$	$(\|u^*(x) - \hat{u}_1(x)\| \leqq)\ x^4 D(x)$
0	0,0	0,0	0,0	0,0
0,1	1,11	0,00152	1,111444	0,000020
0,2	1,24	0,0142	1,252667	0,000381
0,3	1,39	0,0589	1,43757	0,00246
0,4	1,56	0,189	1,6880	0,0108
0,5	1,75	0,75	2,0417	0,0431
0,6	1,96	—	2,572	0,294

Benutzt man die Funktion

$$\hat{u}_1 = \frac{1}{1-x} + \frac{x^3}{3} \qquad \text{(Tabelle 4, Spalte 4)}, \tag{4.4}$$

eine Summe von Lösungen der Differentialgleichungen

$$u' = u^2 \quad \text{und} \quad u' = x^2,$$

so bekommt man mit dem Ansatz $\sigma = D x^4$ ($D = \text{const}$) in entsprechender Weise die Fehlerabschätzung

$$|u^*(x) - \hat{u}_1(x)| \leqq x^4 \cdot D(x) \quad \text{für} \quad 0 \leqq x \leqq \alpha_3 \approx 0{,}6110 \qquad \text{(Tabelle 4, Spalte 5)}.$$

Man beachte, daß $u^* \geqq \frac{1}{1-x}$ ist $\left(\frac{1}{1-x}\right.$ löst die Anfangswertaufgabe $u' = u^2$, $\left.u(0) = 1\right)$, so daß die Lösung $u^*(x)$ eine Singularität für einen Wert $x = x_0 \leqq 1$ besitzt, über den man mit der Fehlerabschätzung nicht hinauskommen kann.

4.2. Eine Klasse partieller Differentialgleichungen. Es sei $\mathfrak{B}$ das aus den Punkten $x = x_1, x_2$ mit

$$a \leqq x_1 \leqq b, \qquad h(a) \leqq x_2 \leqq h(x_1)$$

bestehende Gebiet der x_1, x_2-Ebene. Dabei bedeute $h(x_1)$ eine auf einem Intervall $a \leqq x_1 \leqq b$ erklärte Funktion mit stetiger Ableitung $h'(x_1) > 0$.

Gesucht ist eine auf $\mathfrak{B}$ stetige und mit stetigen Ableitungen $u_{x_1}, u_{x_2}, u_{x_1 x_2}$ versehene Funktion $u(x) = u(x_1, x_2)$, welche die Gleichungen

$$(M[u] =) - u_{x_1 x_2} = f(x_1, x_2, u) \qquad \text{auf } \mathfrak{B},$$

$$(U_1[u] =) \, u(x_1, x_2) = \gamma_1(x_1), \qquad (U_2[u] =) \, u_{x_1}(x_1, x_2) = \gamma_2(x_1) \qquad \text{für} \quad x_2 = h(x_1)$$

erfüllt, wobei die $\gamma_i(x_1)$ gegebene stetige Funktionen bedeuten und $f(x, y) = f(x_1, x_2, y)$ die in Nr. 1.1 geforderten Eigenschaften besitze.

Die gegebene Aufgabe ist der Integralgleichung

$$u(x_1, x_2) = \gamma_1\big(H(x_2)\big) + \int_{H(x_2)}^{x_1} \gamma_2(\xi_1)\, d\xi_1 + \int_{H(x_2)}^{x_1} \int_{x_2}^{h(\xi_1)} f\big(\xi_1, \xi_2, u(\xi_1, \xi_2)\big)\, d\xi_2\, d\xi_1$$

äquivalent, wenn H die Umkehrfunktion von h bedeutet. Diese Integralgleichung läßt sich in der Form (2.6) mit

$$g(x) = g(x_1, x_2) = \gamma_1\big(H(x_2)\big) + \int_{H(x_2)}^{x_1} \gamma_2(\xi_1)\, d\xi_1$$

und

$$G(x, \xi) = G(x_1, x_2; \xi_1, \xi_2) = \begin{cases} 1 & \text{für} \quad H(x_2) \leqq \xi_1 \leqq x_1, \; x_2 \leqq \xi_2 \leqq h(x_1), \\ 0 & \text{sonst auf } \mathfrak{B} \end{cases}$$

schreiben. Die Funktionen $g(x)$ und $G(x, \xi)$ haben die in Nr. 2.1 geforderten Eigenschaften.

Die Vergleichsaufgabe lautet hier

$$-\sigma_{x_1 x_2} \geqq \tilde{f}(x_1, x_2; \sigma + \sigma_1 + \varrho_0) - \tilde{f}(x_1, x_2; \varrho_0) \qquad \text{auf } \mathfrak{B},$$

$$\sigma(x_1, x_2) \geqq 0, \qquad \sigma_{x_1}(x_1, x_2) \geqq 0 \quad \text{für} \quad x_2 = h(x_1).$$

Nach Zusatz 2.3 braucht man nur $\sigma \geqq 0$, $\sigma_{x_1} \geqq 0$ zu fordern, da die Aufgabe

$$-u_{x_1 x_2} = 0 \quad \text{auf } \mathfrak{B}; \qquad u = \sigma, \qquad u_{x_1} = \sigma_{x_1} \quad \text{für} \quad x_2 = h(x_1)$$

in diesem Falle eine nichtnegative Lösung z besitzt.

4.3. Beispiel einer partiellen Differentialgleichung.

Als *Beispiel 4* rechnen wir die Aufgabe

$$-u_{x_1 x_2} = x_1(1 + \alpha u) \quad \text{für} \quad 0 \leqq x_1 < \infty, \; 0 \leqq x_2 \leqq x_1,$$

$$u = u_{x_1} = 0 \quad \text{für} \quad x_2 = x_1$$

mit einer positiven Konstanten α. Um ein Problem der in Nr. 4.2 behandelten Art zu bekommen, schränken wir den Bereich, in welchem die Differentialgleichung erfüllt sein soll, zunächst ein auf

$$\mathfrak{B}: \quad a \leqq x_1 \leqq b, \quad a \leqq x_2 \leqq x_1 \quad \text{mit} \quad 0 \leqq a < b. \tag{4.5}$$

Der Ansatz $u_0 \equiv 0$ liefert

$$u_1 = \tfrac{1}{6}(x_1 - x_2)^2 (2x_1 + x_2),$$

$$u_2 = u_1 + \frac{\alpha}{240}(x_1 - x_2)^4 (5x_1^2 + 4x_1 x_2 + x_2^2).$$

Wir schätzen den Fehler der Funktion u_2 ab, indem wir im Satz 2 u_1 statt u_0, u_2 statt u_1 benutzen. Als Majorante läßt sich wegen $f_y = \alpha x_1$ im Gebiet

$$\mathfrak{G}: \quad x_1, x_2 \in \mathfrak{B}, \quad -\infty < y < \infty$$

die Funktion

$$\tilde{f}(x, y) = \tilde{f}(x_1, x_2; y) = \alpha x_1 y$$

verwenden. Mit $\sigma_1 = u_2 - u_1$ lautet dann das Vergleichsproblem

$$-\sigma_{x_1 x_2} - \alpha x_1 \sigma \geqq \frac{\alpha^2}{240} x_1 (x_1 - x_2)^4 (5x_1^2 + 4x_1 x_2 + x_2^2) \quad \text{auf } \mathfrak{B},$$

$$\sigma = \sigma_{x_1} = 0 \quad \text{für} \quad x_2 = x_1.$$

Der einfache Ansatz $\sigma = c(x_2 - x_1)^6$ ($c = \text{const}$) führt auf die Forderung

$$c\big(30 - \alpha x_1 (x_1 - x_2)^2\big) \geqq \frac{\alpha^2}{240} x_1 (5x_1^2 + 4x_1 x_2 + x_2^2) \quad \text{auf } \mathfrak{B}.$$

Im Bereich (4.5) ist

$$x_1 (x_1 - x_2)^2 \leqq b(b-a)^2, \qquad x_1(5x_1^2 + 4x_1 x_2 + x_2^2) \leqq 10 b^3.$$

Gilt $\alpha b (b-a)^3 < 30$, so existiert also eine nichtnegative Lösung

$$c = \frac{\alpha^2 b^3}{24(30 - \alpha b (b-a)^2)}.$$

Da man, um für eine beliebige feste Stelle x_1, x_2 mit $0 \leqq x_2 \leqq x_1$ eine Schranke zu bekommen, diese als Eckpunkt b, a des verwendeten Dreiecks $\mathfrak{B}$ auffassen kann, erhält man die Fehlerabschätzung

$$|u^*(x_1, x_2) - u_2(x_1, x_2)| \leqq \frac{\alpha^2 x_1^3 (x_1 - x_2)^6}{24(30 - \alpha x_1 (x_1 - x_2)^2)}$$

für alle x_1, x_2 mit $0 \leqq x_2 \leqq x_1$ und $\alpha x_1 (x_1 - x_2)^2 < 30$.

Nachtrag während der Korrektur. Bei den hier mitgeteilten Fehlerabschätzungen für Differentialgleichungen werden zwei gemäß (2.7) verknüpfte Funktionen $u_0(x)$ und $u_1(x)$ benutzt. Aus diesen Ergebnissen lassen sich im Falle $G(x, \xi) \geqq 0$ Abschätzungen folgern, für die man nur die Ausgangsnäherung $u_0(x)$ benötigt, welche die gegebenen Randbedingungen nicht zu erfüllen braucht. Statt $u_1(x) - u_0(x)$ werden dabei der Defekt $d = -M[u_0] + f(x, u_0)$ und gewisse „Randgrößen" verwendet. Darüber soll in Kürze noch etwas ausführlicher berichtet werden.

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